HIGH SCHOOL GEOMETRY

McDOUGALL
SHEPPARD

TORONTO
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Grade V

HIGH SCHOOL

GEOMETRY

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PREFACE

The High School Geometry is intended to provide a course in Theoretical Geometry in accord with current scientific and pedagogical thought.

Realizing that the pupil should gain his first geometrical ideas through making accurate drawings and measurements a considerable number of practical exercises are included throughout the text especially in the introduction.

All drawings should be drawn neatly and accurately with geometrical instruments.

Exceptional care has been given to the arrangement of the propositions and their proofs. The theoretical and practical work are co-ordinated as far as possible.

The book contains an abundant supply of carefully selected and graded exercises. Those given in sets throughout the Books will be found suitable for the work of average classes, and about sufficient in number to fix the subject matter in the minds of the pupils. All the propositions contained in the miscellaneous collections at the ends of the Books could be worked through by a few only of the best pupils, and should be used also by the teachers, from time to time, as a store from which to draw suitable material for review purposes.

While simplicity has been constantly kept in mind in the choice of proofs, it should not be assumed that other proofs, just as good, cannot in many cases be given. Students should be constantly encouraged to work out methods of their own, and to keep records of the best methods in their note-books.

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations are used :—

Fig.	Figure.
Const.	Construction.
Hyp.	Hypothesis.
Cor.	Corollary.
e.g.	*exempli gratia*, for example.
i.e.	*id est*, that is.
p.	page.
$\because$	because, since.
$\therefore$	therefore.
rt.	right.
st.	straight.
$\angle$, $\angle$s, $\angle$d	angle, angles, angled.
$\triangle$, $\triangle$s	triangle, triangles.
$\parallel$, $\parallel$s	parallel, parallels.
$\parallel$gm, $\parallel$gms	parallelogram, parallelograms.
sq., sqs.	square, squares.
AB^2	the square on AB.
rect.	rectangle.
AB.CD	the rectangle contained by AB and CD.
AB : CD, or $\frac{AB}{CD}$	the ratio of AB to CD.
$+$	plus, together with.
$-$	minus, diminished by.
$\perp$	is perpendicular to, a perpendicular.
$=$	is equal to, equals.
$>$	is greater than.
$<$	is less than.
$\equiv$	is congruent to, congruent.
$\parallel\!\mid$	is similar to, similar.

CONTENTS

THEORETICAL GEOMETRY

INTRODUCTION

Preliminary Definitions and Explanations

Instruments. The pupil should use a hard sharp pencil. All drawings should be neat and accurate. An accuracy within one or two per cent. can be obtained with ordinary instruments.

The pupil should have:

A ruler graduated in inches and fractions of an inch, and also in centimetres and millimetres.

A pair of compasses.

A protractor. The protractor may be either semicircular or rectangular.

A set-square.

He should also have a supply of tracing paper.

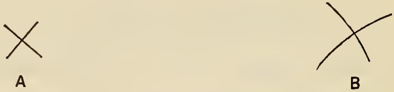
1. The straight line. Fold and crease a sheet of fairly stiff paper.

The crease thus formed is a straight line. Test the accuracy of your ruler by drawing a line with it on paper and fitting the crease of the folded paper to the line.

Points, Lines, and Surfaces

All pupils, beginning the study of geometry, know in a general way what is meant by a **point,** a **line** and a **surface.** In geometry these terms are used in a strict sense which needs some explanation.

2. **The point.** Draw two st. lines cutting each other. The place where they intersect is a **point.** The best way to indicate an exact position on paper is by the intersection of two lines.



A **point** is that which has position but not magnitude. A geometrical point has neither length nor breadth nor thickness. A dot made by a sharp pencil may be taken as roughly representing a point. A point is named by placing beside it a capital printed letter; .A is the **point** A.

3. **A line** is that which has length but neither breadth nor thickness.

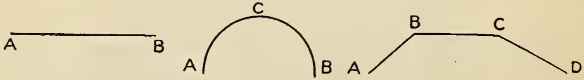
The mark that we use to represent a line has breadth and some small thickness and, consequently, only roughly represents the idea.

A line may be straight, curved or broken.

A **straight line** has the same direction throughout its whole length.

OR

A **straight line** is the shortest distance between two points. AB is a straight line.



A **curved line** changes its direction continually from point to point. ACB is a curved line.

A **broken line** is a line composed of different successive straight lines. **ABCD** is a broken line.

4. A **surface** is that which has length and breadth but no thickness.

A sheet of tissue paper has length and breadth and very little thickness. It thus roughly represents the idea of a surface. In fact the sheet of paper has two well-defined surfaces separated by the substance of the paper.

The boundary between two parts of space is a surface.

Surfaces may be either plane or curved.

The following property distinguishes plane surfaces from curved surfaces and may be used as the definition of a plane surface:—

A **plane surface** is that in which **any** two points being taken, the straight line joining them lies wholly in that surface.

Give examples of curved surfaces on which straight lines may be drawn in certain directions. Notice the force of the word "any" in the definition above.

5. A **solid** is that which has length, breadth and thickness.

6. Any combination of points, lines, surfaces and solids is called a **figure.**

7. **Geometry** is the science which investigates the properties of figures and the relations of figures to one another.

8. In **Plane Geometry** the figure, or figures, considered in each proposition are confined to one plane, while

Solid Geometry treats of figures the parts of which are not all in the same plane.

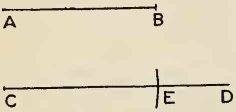
Plane Geometry is also called Geometry of Two Dimensions (length and breadth), and Solid Geometry is called Geometry of Three Dimensions (length, breadth and thickness).

9. **Measurement of straight lines.** It is required to measure the distance between the two points A and B.

.A .B

Set one point of the compasses on A and the other on B and, holding them so as not to change the setting, place the points on the graduated ruler with one of them at the zero mark. Read the number shown by the other.

10. **From the greater of two straight lines cut off a part equal to the less.**



Draw two st. lines AB, CD, of which CD is the greater.

Set the compasses at the distance AB.

Place the steel point at C and make a short mark cutting CD at E.

The part CE has been cut off = AB.

11.—Exercises

1. Draw a st. line, guess its middle point and mark it. Test the accuracy of your guess by measuring the parts.

2. Draw a st. line 7·8 cm. in length. Cut off from one end a part = 3·9 cm. and measure the remainder to see if you have found the middle point exactly.

3. From a given point draw a st. line that is bisected at another given point.

4. Draw a st. line **BC** and mark a point **A** not in **BC**. Using the compasses and ruler draw a st. line 4 cm. in length having one end at **A** and the other in **BC**.

If the point **A** is not too far from **BC**, two such lines can be drawn.

5. Draw two st. lines cutting each other at the point **A**. In each of the lines find two points each of which is 6 cm. from **A**.

6. Mark a point **A** on a sheet of paper. It is required to find a point 5 cm. from **A**. Can there be more than one such point? If so, how many? Where are they situated? Using the compasses draw the curved line that passes through all points on the paper that are 5 cm. from **A**. This curved line is called the **circumference of a circle**. A part of a circumference is called an **arc**.

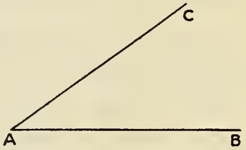
7. Draw a st. line **AB**, 6 cm. in length. It is required to find a point 5 cm. from **A** and 4 cm. from **B**. Draw an arc through points each 5 cm. from **A** and another through points each 4 cm. from **B**. The two arcs cut at a point **C** which is 5 cm. from **A** and 4 cm. from **B**. How many such points are there? Join **C** to **A** and **B**. The figure **ABC** has three sides and is called a **triangle**.

8. Make a △ having each side = 6 cm.

9. Make a △ having its sides 47, 58 and 74 mm.

Angles

12. **Definitions.**—Let a straight line **AB** rotate about **A**, in the plane of the paper, from a position **AB** to another position **AC**. The amount of turning which the line has done in rotating about **A** from the position **AB** to the position **AC** is called an **angle**.

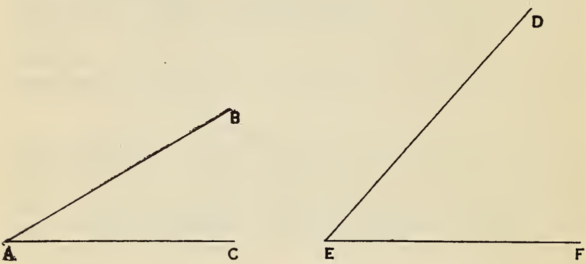


The point **A** is called the **vertex** of the angle.

The lines **AB** and **AC** are called the **arms** of the angle.

The angle in the figure may be called the angle BAC, or the angle CAB. The letter at the vertex must be the middle one in reading the angle.

The single letter at the vertex is sometimes used to denote the angle when there can be no doubt as to which angle is meant.



13. Draw two $\angle$s BAC, DEF.

Make a tracing of $\angle$ BAC and mark the tracing with the same letters, B, A, C.

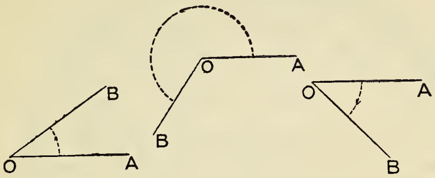
Place the tracing on $\angle$ DEF so that the point A is exactly on E and the line AC falls along EF.

If AB falls between EF and ED, the $\angle$ BAC is less than the $\angle$ DEF.

Make another $\angle$ DEF such that when the tracing of BAC is placed on DEF as before AB falls beyond ED. The $\angle$ BAC is greater than this second $\angle$ DEF.

If the tracing of AB falls exactly on ED, the $\angle$ BAC = $\angle$ DEF.

14. Suppose a straight line **OB** to be fixed, like a rigid rod on a pivot at the point **O**, and be free to rotate in the plane of the paper.



If the line **OB** start from any position **OA**, it may rotate in either of two directions—that in which the hands of a clock rotate, or in the opposite.

When **OB** starts from **OA** and stops at any position an angle is formed with **O** for its vertex and **OA** and **OB** for its arms.

15. An angle is said to be positive or negative according to the direction in which the line that traces out the angle is supposed to have rotated. The direction contrary to that in which the hands of a clock rotate is commonly taken as positive.

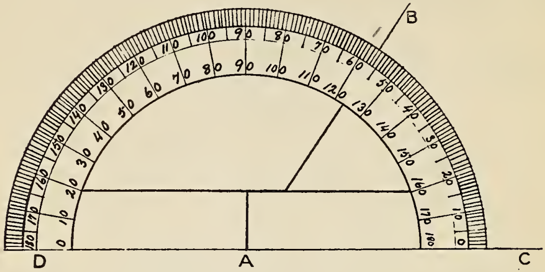
16. The magnitude of an angle depends altogether on the amount of rotation, and is quite independent of the lengths of its arms.

17. **Use of the protractor.** The protractor is used to:

(*a*) Measure a given $\angle$ in degrees;

(*b*) Make an $\angle$ containing a given number of degrees.

18. It is required to find the number of degrees in a given ∠ BAC.



Place the protractor so that its centre is on the vertex A of the given ∠ and its zero line is exactly along AC. The position of the line AB shows the number of degrees on the protractor. In the diagram the ∠ BAC is seen to equal 57°. At the same time, if CA is produced to D, the number of degrees in the ∠ BAD is seen to be 123.

19. It is required to make an ∠ of 32°.

Draw a st. line AC on paper and place the protractor so that its centre is at A and its zero line is exactly along AC. Hold the protractor in place and make a mark on the paper at the line for thirty-two degrees of the protractor. Remove the protractor and draw a line from A to the mark.

An ∠ of, say, 35° can be set off at a point A in a st. line AC without using the ruler. Arrange the protractor so that the middle point of its base is on A and the 35° mark is on CA produced through A. A line drawn along the base of the protractor makes the required ∠ with AC.

20.—Exercises

1. Draw on paper a number of angles and use the protractor to find the number of degrees in each.

2. Make angles of 15°, 45°, 79°, 152°, 165°.

3. Make successive angles **BAC, CAD, DAE, EAF** of 17°, 42°, 13°, 18° respectively. Measure angle **BAF**. What do you observe regarding the position of **AF**?

4. Make successive angles **BAC, CAD, DAE** of 60°, 70°, 50° respectively. Measure angle **BAE**. What do you observe regarding the position of **AB** and **AE**?

5. At the ends of the line **AB** make the angles **BAC** and **ABC** respectively equal to 30° and 60°. Measure the angle **ACB**. Find the sum of the angles **A, B** and **C**.

6. Draw two straight lines **ABC, DBE** cutting at **B**. Measure the angles **ABE, DBC**. Find their sum.

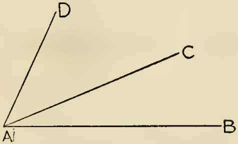
7. Through what angle does the earth rotate round its axis in one hour? Angle 15°

8. Through what angle does the minute hand of a clock move in forty-five minutes? 75°

9. How long does it take the minute hand of a clock to turn through 90°, 135°, 180°, 360°?

10. What is the angle between the hands of a clock at (a) 3 p.m.; (b) 6 p.m.; (c) 10 a.m.?

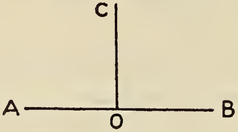
21. Definition.—When two angles have the same vertex and a common arm, and the remaining arms on opposite sides of the common arm, they are said to be **adjacent angles.**



Thus **BAC** and **CAD** are adjacent angles having the same vertex **A** and the common arm **AC**.

But angles **BAD** and **CAD**, with the same vertex and the common arm **AD** are not adjacent angles.

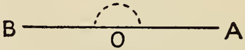
22. When one straight line stands on another so as to make the adjacent angles equal to one another each of the angles is called a **right angle** and each line is said to be **perpendicular** to the other.



If $\angle$ **AOC** = $\angle$ **COB** then each $\angle$ is a right angle and **OC** is perpendicular to **AB**.

23. 1 right $\angle$ = 90 degrees (°)
1 degree = 60 minutes (′)
1 minute = 60 seconds (″)

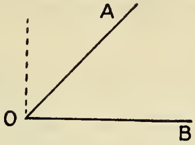
24. If one arm **OB** of an angle turns until it makes a straight line with the other arm **OA**, the angle so formed is called a **straight angle.**



AOB is a straight angle.

A straight $\angle$ = 2 right angles.
= 180°.

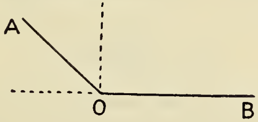
25. An angle which is less than one right angle is said to be **acute.**



Thus, an acute angle is less than 90°.

AOB is an acute angle.

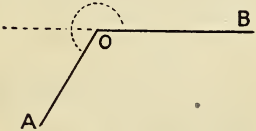
26. An angle which is greater than one right angle but less than two right angles is said to be **obtuse.**



Thus, an obtuse angle lies between 90° and 180°.

AOB is an obtuse angle.

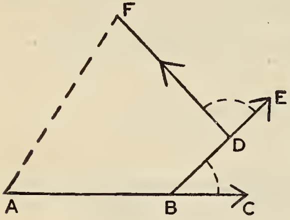
27. An angle which is greater than two right angles but less than four right angles is said to be **reflex.**



Thus, a reflex angle lies between 180° and 360°.

AOB is a reflex angle.

28. A man walks 2 miles, turns through 45° to his left, goes one mile, turns through 90° to his left and goes 2 miles farther. Show that he is nearly $2\frac{1}{2}$ miles from where he started.



Take half-an-inch to represent a mile. Make **AB** = one inch. Make $\angle$ **CBE** = 45°. Make **BD** = half-an-inch. Make $\angle$ **EDF** = 90°. Make **DF** = one inch.

Measure **AF** in half inches.

29.—Exercises.

1. A man walks 5 miles north and then turns to the east. How far must he walk in this direction to be 13 miles in a direct line from the starting point?

2. Two men start from the same point. One walks 3 miles north and turning to the east walks $5\frac{1}{2}$ miles farther. The other walks $6\frac{1}{2}$ miles west and then 4 miles south. Find how far apart the men now are.

3. A 51-foot ladder, whose foot is 24 ft. from the bottom of a wall just reaches a window. Find the height of the window from the ground.

4. A man goes from **A** 5 miles east to **B**, from **B** 6 miles northeast to **C**, from **C** 7 miles west to **D**, and from **D** 4 miles northwest to **E**. Find the direct distance from **A** to **E**.

(Answers.—1. 12 miles; 2. 13·9 miles nearly; 3. 45 ft.; 4. 7 miles nearly.)

30. **Definitions.**—When two angles are such that their sum is two right angles, they are said to be **supplementary** angles, or each angle is said to be the **supplement** of the other.

If two $\angle$s are equal, their supplementary $\angle$s are also equal.

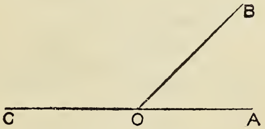
When two angles are such that their sum is one right angle, they are said to be **complementary** angles, or each angle is said to be the **complement** of the other.

The complements of equal angles are also equal.

31.—Exercises

1. What is the supplement of 27°?
2. What is the supplement of 142°?
3. What is the complement of 73°?
4. What angle is equal to its supplement?
5. What angle is equal to its complement?
6. What angle is equal to half its supplement?
7. What angle is equal to half its complement?
8. What is the complement of a right angle?
9. What is the supplement of a right angle?

32. Let a st. line starting from OA revolve through two successive $\angle$s



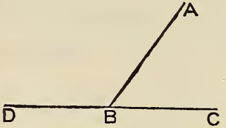
AOB, BOC such that OC is in the same st. line with OA, but in the opposite direction from the point O, and consequently AOC is a st. $\angle$.

$$\because \angle \mathsf{AOB} + \angle \mathsf{BOC} = \text{the st. } \angle \mathsf{AOC},$$

$$\therefore \angle \mathsf{AOB} + \angle \mathsf{BOC} = 2 \text{ rt. } \angle\text{s}.$$

Thus the angles which one straight line makes with another on the same side of that other are together equal to two right angles.

33. Let the adjacent $\angle$s ABC, ABD be together equal to two rt. $\angle$s.



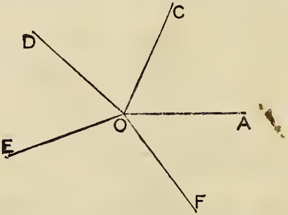
$$\angle \mathsf{ABD} + \angle \mathsf{ABC} = \text{two rt. } \angle\text{s} = \text{a st. } \angle.$$

That is, $\angle$ DBC is a st. $\angle$,

and $\therefore$ line DBC is a st. line.

Thus, if two adjacent angles are together equal to two right angles, the exterior arms of the angles are in the same straight line.

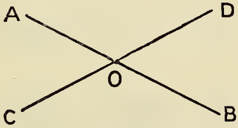
34. Let a st. line OB, starting from the position OA, and rotating in the positive direction, trace out the successive $\angle$s: AOC, COD, DOE, EOF, FOA.



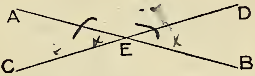
The sum of the successive $\angle$s is a complete revolution, and therefore equal to four rt. $\angle$s.

Thus, if any number of straight lines meet at a point, the sum of the successive angles is four 360° **right angles.**

35. When two straight lines such as **AB**, **CD** cross one another at **O** the angles **COA**, **BOD** are said to be **vertically opposite.** The angles **COB**, **AOD** are also vertically opposite.



36. **Each of the angles formed by two intersecting straight lines is equal to the vertically opposite angle.**



The two st. lines **AB**, **CD** cut each other at **E**.

Prove that (1) ∠ **AEC** = ∠ **BED**,

(2) ∠ **AED** = ∠ **BEC**.

Proof.—∵ **CED** is a st. line,

∠ **AEC** + ∠ **AED** = two rt. ∠s.

∵ **AEB** is a st. line,

∠ **AED** + ∠ **DEB** = two rt. ∠s.

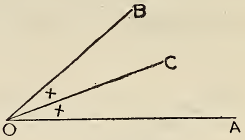
∴ ∠ **AEC** + ∠ **AED** = ∠ **AED** + ∠ **DEB**.

From each of these equals take away the common ∠ **AED** and the remainders must be equal to each other.

∴ ∠ **AEC** = ∠ **DEB**.

In the same manner it may be shown that ∠ AED = ∠ CEB.

37.



If a straight line revolve in the positive direction about the point O from the position OA to the position OB, it must pass through some position OC such that ∠ AOC = ∠ COB.

A straight line which divides an angle into two equal angles is called the **bisector** of the angle.

When a construction is represented in a diagram, although it has not previously been proved that it can be made, it is called a **hypothetical construction.** Thus OC has been drawn to represent the bisector of ∠ AOB.

38.—Exercises

1. If one of the four ∠s made by two intersecting st. lines be 17°, find the number of degrees in each of the other three.

2. Two st. lines ABD, CBE cut at B, and ∠ ABC is a rt. ∠. Prove that the other ∠s at B are also rt. ∠s.

3. If in the figure of Art. (36) the ∠ AEC = $\frac{2}{3}$ ∠ AED, find the number of degrees in each ∠ of the figure.

4.

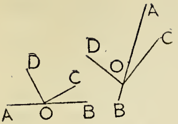


Fig. 1 Fig 2

DOC is a rt. $\angle$, and through the vertex **O** a st. line **AOB** is drawn.

Prove that:—

In Fig. 1, $\angle$ **BOC** + $\angle$ **AOD** = a rt. $\angle$.

In Fig. 2, $\angle$ **BOC** — $\angle$ **AOD** = a rt. $\angle$.

5. In the diagram,

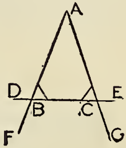
$\angle$ **ABC** = $\angle$ **ACB**.

Prove that

(1) $\angle$ **ABD** = $\angle$ **ACE**,

(2) $\angle$ **FBC** = $\angle$ **GCB**,

(3) $\angle$ **DBF** = $\angle$ **ECG**.



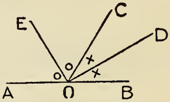
Test these results by drawing the diagram with $\angle$s **ABC**, **ACB** each = 75°.

6. In the diagram,

AOB is a st. line,

$\angle$ **COD** = $\angle$ **DOB** and

$\angle$ **AOE** = $\angle$ **EOC**.

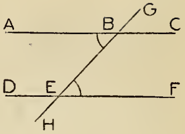


Prove that **EOD** is a rt. $\angle$, and that $\angle$ **AOE** is the complement of $\angle$ **BOD**.

7. **E** is a point between **A** and **B** in the st. line **AB**; **DE**, **FE** are drawn on opposite sides of **AB** and such that $\angle$ **DEA** = $\angle$ **FEB**. Show that **DEF** is a st. line.

8. Four st. lines, **OA**, **OB**, **OC**, **OD**, are drawn in succession from the point **O**, and are such that $\angle$ **AOB** = $\angle$ **COD** and $\angle$ **BOC** = $\angle$ **DOA**. Show that **AOC** is a st. line, and also that **BOD** is a st. line.

9. In the diagram, **ABC, DEF, GBEH** are st. lines and $\angle$ **ABE** = $\angle$ **BEF.**



Prove that

(1) $\angle$ **CBE** = $\angle$ **BED,**
(2) $\angle$ **GBC** = $\angle$ **DEH,**
(3) $\angle$ **ABG** = $\angle$ **BED,**
(4) $\angle$s **CBE, BEF** are supplementary,
(5) $\angle$s **ABE, BED** are supplementary.

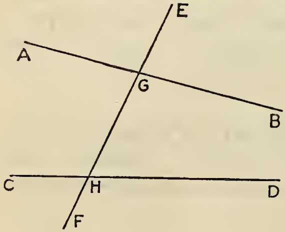
Test these results by drawing the diagram with the $\angle$s **ABE, BEF** each = 70°.

Parallel Lines

39. **Parallel straight lines** are those which have the same direction throughout their entire length.

OR

Two straight lines in the same plane which do not meet when produced for any finite distance in either direction are said to be **parallel** to one another.



40. A straight line which cuts two, or more, other straight lines is called a **transversal.**

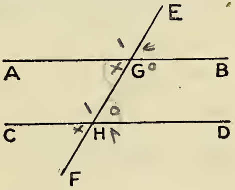
Draw a st. line **EF** cutting two other st. lines **AB** and **CD** at **G** and **H.**

Eight $\angle$s are thus formed, four of which, AGH, BGH, CHG, DHG, being between AB and CD, are called **interior** $\angle$s. The other four are called **exterior** $\angle$s.

The interior $\angle$s AGH and GHD, on opposite sides of the transversal, are called **alternate** $\angle$s. Thus also, BGH and GHC are alternate $\angle$s.

Name four pairs of equal angles in the diagram.

41. Since parallel straight lines have the same direction they each **deviate** by the same amount from any other direction; that is, they make the same angle with any other line which intersects them.



Hence, if EF cuts the two parallel lines AB and CD at G and H, then $\angle$EGB = $\angle$GHD.

i.e. **If a transversal cut two parallel lines the exterior angle equals the interior and opposite angle on the same side of the transversal.**

Also, if $\angle$ EGB = $\angle$ GHD the lines AB and CD deviate by the same amount from the direction FE. Hence, AB and CD have the same direction. $\therefore$ AB is parallel to CD.

Hence, **if a transversal cut two other straight lines and makes the exterior angle equal to the interior and opposite on the same side of the transversal the two straight lines are parallel.**

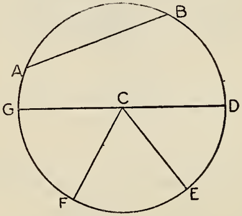
42.—Exercises

1. By means of parallel rulers draw two parallel straight lines AB and CD. Draw any transversal EF cutting AB and CD at G and H. Measure the angle EGB and the angle GHD. Are they equal? Measure the angles AGH and GHD; are they equal?

2. A transversal EF cuts two parallel straight lines AB and CD at G and H. Write out twelve pairs of equal angles.

The Circle

43. A **circle** is a PLAIN figure bounded by a curved line, called the **circumference**, and is such that all straight lines drawn from a certain point within the figure, called the **centre**, to the circumference are equal to one another.



In a circle a st. line drawn from the centre to the circumference is called a **radius.** (Plural—radii.)

A st. line, as AB, joining two points in the circumference is called a **chord.**

If a chord passes through the centre, as GD, it is called a **diameter.**

A part of the circumference, as the curved line FED, is called an **arc.**

A line drawn from a point in one arm of an angle to a point in the other arm is said to **subtend** the angle.

In the diagram the arc FE subtends the ∠ FCE; or in any △ each side subtends the opposite ∠.

A semi-circle is the figure bounded by a diameter and the part of the circumference cut off by the diameter.

Figure
The arc GBD is a semi-circle.

Rectilineal Figures

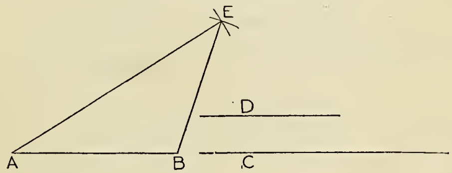
44. **Definitions.**—A figure formed by straight lines is called a **rectilineal** figure.

The figure formed by three straight lines which intersect one another is called a **triangle.**

The three points of intersection are called the **vertices** of the triangle.

The lines between the vertices of the triangle are called the **sides** of the triangle.

45. **Construct a triangle with sides of given length.**



Let AB, C and D be the given lengths.

Construction.—With centre A and radius C describe an arc.

With centre B and radius D describe an arc cutting the first arc at E.

Join **EA, EB.**

AEB is the required △.

Question—*In what case would the above construction fail?*

46.—Exercises

1. Construct a triangle whose sides are 3, 4 and 5 inches.

2. One side of a triangle is 2 inches long. The angles at the ends of this side are 30° and 60°. Construct the triangle.

3. One side of a triangle is 6 cms. long. The angles at the ends of this side are 135° and 10°. Construct the triangle.

4. We wish to find the sum of the three ∠s of a △. Make a △ with its sides 2, 3 and 4 inches; measure its ∠s and add the results. Do the same with several △s of different shapes. Is the sum always equal to a st. ∠?

Draw a △ and cut it out; mark the three ∠s, cut them off and place them in a st. line.

5. Construct the △, $b = 3$ cm., $c = 5$ cm. and $\mathsf{A} = 23°$. Measure a. (Ans. 25 mm.)

(Note.—In a △ **ABC** the capital letters **A, B, C** are used to represent the size of the respective ∠s and the small letters a, b, c to represent the length of the sides respectively opposite to the ∠s **A, B, C.** Thus a is the length of **BC,** b of **CA** and c of **AB.**)

6. Construct the △ having $a = 13$ cm., $b = 7$ cm. and $\mathsf{C} = 60°$. Measure **A** and **B.** (Ans. $\mathsf{A} = 87\frac{1}{2}°$, $\mathsf{B} = 32\frac{1}{2}°$.)

7. Construct the △ having $a = 84$ mm., $b = 42$ mm., $\mathsf{C} = 120°$. Measure c, **A** and **B.** (Ans. $c = 111$ mm., $\mathsf{A} = 41°$, $\mathsf{B} = 19°$ nearly.)

8. The sides of a △ are 56, 65 and 33 mm. Measure the greatest ∠. (Ans. 90°.)

9. The sides of a △ are 32, 40, 66 mm. Measure the greatest ∠. (Ans. $132\frac{1}{2}°$ nearly.)

10. Draw two st. lines **ABC, DBE** cutting at **B.** Compare the ∠s **ABE, CBD.** Are they equal? Is the ∠ **ABD** = ∠ **CBE?** Use tracing paper.

11. Draw a △ **ABC** having $b = c$. Compare the ∠s **B, C.** Are these ∠s equal. Use tracing paper.

12. Draw a △ having its three sides equal to each other. How many degrees are there in each of its ∠s?

13. Draw a △ **ABC** having **A** = 35° and $b = 2c$. Is **B** = 2**C**?

14. Draw a △ **ABC** and produce **BC** to **D.** Measure ∠s **A, B, ACD.** Is the exterior ∠ **ACD** equal to the sum of ∠s **A** and **B**?

15. Make two △s **ABC, DEF** having the same number of degrees in the ∠s **A** and **D** and having also **AB** = **DE** and **AC** = **DF.** Make a tracing of the △ **ABC** and mark it with the same letters **A, B, C** at the corresponding ∠s. Can this tracing be made to fit the △ **DEF** with **A** on **D**? If so, what ∠s of the △s are seen to be equal? What about the sides **BC** and **EF**?

16. Make two △s **ABC, DEF** having **AB** = **DE, BC** = **EF,** and **AC** = **DF.** Make a tracing of **ABC** and fit it to **DEF.** What ∠s are found to be equal? How are these ∠s related to the sides that were made equal?

17. Construct two △s **ABC, DEF** having **BC** = **EF,** ∠ **ABC** = ∠ **DEF,** ∠ **ACB** = ∠ **DFE.** Make a tracing of **ABC** and fit it to **DEF.** What other angles and sides are equal? Are their areas equal?

18. Construct a triangle **ABC** on the base **BC** so that the ∠ **B** = 40°, ∠ **C** = 40°.

(*a*) Measure size of ∠ **A.**

(*b*) Find the sum of ∠ **A** + ∠ **B** + ∠ **C.**

(*c*) Compare the lengths of **AB** and **AC.**

19. Construct a triangle **ABC** on the base **BC** so that ∠ **B** = 60°, ∠ **C** = 60°.

(*a*) Measure the size of ∠ **A.**

(*b*) Find the sum of ∠ **A** + ∠ **B** + ∠ **C.**

(*c*) Compare the lengths of the sides of the triangle.

20. Construct a triangle **ABC** on the base **BC** so that ∠ **B** = 37°, ∠ **C** = 64°.

(*a*) Measure the size of ∠ **A.**

(*b*) Find the sum of $\angle \mathsf{A} + \angle \mathsf{B} + \angle \mathsf{C}$.

(*c*) Compare the lengths of the sides of the triangle.

47. Triangles may be classified according to their **sides** or their **angles.**

A triangle is said to be:

Equilateral when all its sides are equal.

Isosceles when two of its sides are equal.

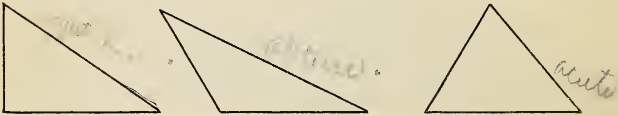
Scalene when its sides are all unequal.

A triangle is said to be:

Right-angled when one of its angles is a right angle.

Obtuse-angled when one of its angles is obtuse.

Acute-angled when all three of its angles are acute.



In a right-angled triangle the side opposite to the right angle is called the **Hypotenuse.**

48.—Exercises

1. Construct an equilateral triangle and measure the size of each of its angles. What is the sum of all of its angles?

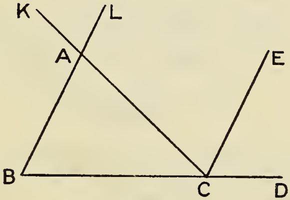
2. Construct an isosceles triangle whose equal sides are each 2 inches long. Measure the size of each angle. What is the sum of all of its angles?

3. Construct a scalene triangle. Measure each of its angles. What is the sum of all of its angles?

4. Construct (*a*) a right-angled isosceles triangle.
(*b*) a right-angled scalene triangle.
(*c*) an acute-angled isosceles triangle.
(*d*) an acute-angled scalene triangle.
(*e*) an obtuse-angled isosceles triangle.
(*f*) an obtuse-angled scalene triangle.

5. On a given base describe an isosceles triangle having each of the equal sides double the base.

49. The sum of the angles of a triangle is two right angles.



Let ABC be any triangle.

Then $\angle$ ABC + $\angle$ ACB + $\angle$ BAC = 2 right angles.

Produce BC to D, CA to K, BA to L.

Suppose CE to be parallel to AB.

Then $\angle$ KAL = $\angle$ ACE. (Int. 41)

But $\angle$ BAC = $\angle$ KAL. (Int. 36)

$\therefore$ $\angle$ BAC = ACE. (1)

Also $\angle$ ABC = $\angle$ ECD. (2)

and $\angle$ ACB = $\angle$ ACB.

Adding $\angle$ BAC + $\angle$ ABC + $\angle$ ACB = $\angle$ ACB + $\angle$ ACE + $\angle$ ECD.

= a straight angle.

= 2 right angles.

Hence the sum of the angles of a triangle is two right angles.

Also, adding (1) and (2)

$\angle$ BAC + $\angle$ ABC = $\angle$ ACE + $\angle$ ECD,

= $\angle$ ACD.

∴ the exterior angle of a triangle is equal to the sum of the two interior and opposite angles and, therefore, is greater than either of them.

Since the three angles of any triangle are together equal to two right angles, then any two angles of a triangle are together less than two right angles.

50.—Exercises

1. The sides of a triangular plot of ground are 100 feet, 200 feet, 250 feet respectively. Using a scale of 100 feet to the inch construct a triangle to represent the plot and measure the size of its angles.

2. A ladder 20 feet long is placed against the side of a house. The foot of the ladder is 8 feet from the side of the house. Find the size of the ang'e between the ladder and the house. How could you find the size of the angle between the ladder and the ground without measuring it?

3. What must be the length of a ladder set at an angle of 75° with the ground to reach a window 24 feet high?

4. The top of a flag pole, broken by the wind, falls so that it touches the ground at a distance of 20 feet from the foot of the pole and is inclined to the ground at an angle of 65°. What is the height

of the portion that remains standing and what is the total length of the pole?

5. A guy rope 50 feet long is attached to the top of a mast and makes an angle of 55° with the level of the ground. Find the height of the mast.

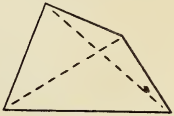
6. The distance between two objects A, B cannot be measured directly by the use of the chain or tape because of an intervening body of water. To find the distance from A to B a third point C is chosen from which A and B are visible and the following measurements are made: AC = 300 ft., ∠ CAB = 50°, ∠ ABC = 70°. By drawing a figure to scale find the distance AB.

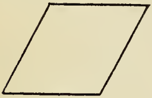
7. Two points A and B are separated by a body of water. In order to find the distance between them another point C is taken which is visible from both A and B. The following measurements are made: AC = 600 feet, ∠ BAC = 50°, ∠ ACB = 70°. Find the distance from A to B.

Quadrilaterals

51. A **quadrilateral** is a plane figure bounded by four straight lines.

The lines which join opposite corners of a quadrilateral are called **diagonals.**





A **rhombus** is a quadrilateral that has all its sides equal.

A **square** is a quadrilateral that has all its sides equal and one angle is a right angle.

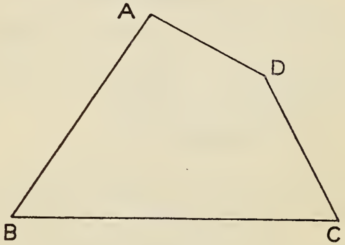
A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

A **rectangle** is a parallelogram which has one of its angles a right angle.

(It will be proved later that all the angles of a rectangle are right angles.)

A **trapezium** is a quadrilateral which has one pair of parallel sides.

52. To construct a quadrilateral given three sides and the two included angles.



Suppose we are to construct a quadrilateral whose side AB = 1.5 inches, BC = 2 inches, CD = 1 inch, and $\angle$ B = 55°, $\angle$ C = 62°.

Construction:

Draw BC = 2 inches.

Make the $\angle$ CBA = 55°. (Using a protractor.)

Mark off a length BA = 1.5 inches.

Make $\angle$ **BCD** $= 62°$.

Mark off a length **CD** $=$ 1 inch.

Join **AD**.

Then **ABCD** is the required quadrilateral.

53.—Exercises

1. Construct a $\triangle$ having given two sides and the included $\angle$.

2. Construct a $\triangle$ having given two $\angle$s and the side between them.

3. Construct a $\triangle$ having given two $\angle$s and the side opposite.

4. Construct a $\triangle$ having given two sides and the $\angle$ opposite to one of them.

5. Construct a quadrilateral given the four sides and one diagonal.

6. Construct a quadrilateral given the four sides and one $\angle$.

7. Prove that if two $\triangle$s have two $\angle$s of one respectively equal to two $\angle$s of the other, the third $\angle$ of one is equal to the third $\angle$ of the other.

8. Show that the sum of the $\angle$s of a quadrilateral is equal to four rt. $\angle$s.

9. Show that the sum of the $\angle$s of a pentagon is six rt. $\angle$s.

10. Construct two squares whose sides are one inch, two inches. Compare their areas.

11. Construct a rhombus whose side is 2 inches and one $\angle$ 60°. Measure the size of the opposite $\angle$s.

12. Draw any $\|$gm **ABCD** with its diagonals cutting in **E**.

(*1*) Measure $\angle$ **A**, $\angle$ **B**, $\angle$ **C**, $\angle$ **D**. What relations do you find?

(*2*) Measure **AE**, **BE**, **CE**, **DE**. What relations do you find?

13. Draw a rectangle whose sides are two inches and one inch. Measure each diagonal.

14. Construct any rhombus **ABCD**. Let the diagonals intersect at **E**.

(*1*) Measure **AE, BE, CE, DE.** What relation do you find?

(*2*) Measure the $\angle$s at **E.**

(*3*) Measure the $\angle$s at **A, B, C** and **D** which the diagonals make with the sides.

15. Construct a trapezium whose parallel sides are 2 inches and 3 inches respectively. What do you notice regarding the number of trapeziums you can draw with this data?

54. A **polygon** is a plane figure contained by more than four straight lines.

An **equilateral polygon** has all its sides equal.

An **equiangular polygon** has all its angles equal.

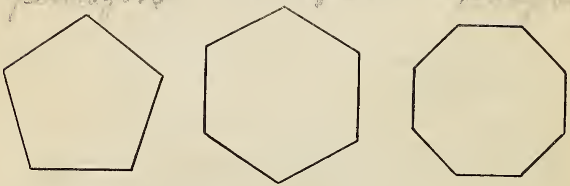
A **regular polygon** is both equilateral and equiangular.

A **pentagon** is a polygon bounded by five sides.

A **hexagon** is a polygon bounded by six sides.

An **octogon** is a polygon bounded by eight sides.

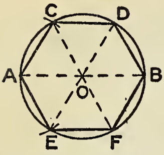
pentagon Hexagon octagon



55.—Exercises

1. Describe a circle of radius one inch. Through the centre **O** draw two diameters **AOB, COD** at right angles. Join **AC, CB, BD, DA.** Measure the sides and angles of the figure **ACBD.** What kind of quadrilateral is it?

2. Draw a circle and through its centre draw the st. line **AOB**.



With centre **A** and the same radius mark the points **C, E** on the circumference.

Draw **CO, EO** and produce them to meet the circumference at **F, D**. Draw **AC, CD, DB, BF, FE** and **EA**.

Are all the sides and all the $\angle$s of this six-sided figure equal? What kind of polygon is **ACDBFE**? Are the $\triangle$s which fill in the space about the point **O** all equilateral?

Geometrical Reasoning

56. Two general methods of investigating the properties or relations of figures may be distinguished as the Practical Method and the Theoretical Method.

By the Practical Method some properties may be tested by measurement, paper-folding, etc.

By the Theoretical Method it may be shown theoretically that the property follows as a necessary result from others that are already known to be true.

The Theoretical Method has certain advantages over the Practical Method. Measurements, etc., are never exact, and in many cases cannot be made directly; but in the Theoretical Method, starting from certain simple statements, called **axioms**, the truth of which is self-evident, or, it may be in some cases, assumed, the consequent statements follow with absolute certainty.

The Practical Method is also known as the Inductive Method of Reasoning, and the Theoretical Method as the Deductive Method.

57. Figures may be compared by making a tracing of one of them and fitting the tracing on the other. In many cases the process may be made a mental

operation and the comparison made with absolute certainty by means of the following axiom:—

A figure may be, actually or mentally, transferred from one position to another without change of form or size.

When two figures are shown to be exactly equal in all respects by supposing one to be made to fit exactly on the other, the proof is said to be by the **method of superposition.**

Figures which exactly fill the same space are said to **coincide** with each other.

58. Figures that are equal in all respects, so that one may be made to fit the other exactly, are said to be **congruent.**

The sign $\equiv$ is used to denote the congruence of figures.

59. In general, a **proposition** is that which is stated or affirmed for discussion.

Propositions

60. The propositions in geometry are of two kinds, **Theorems** and **Problems.**

A **theorem** is the statement of some geometrical truth. For example:—If two straight lines cut each other the vertically opposite angles are equal.

A **problem** is the statement of some required geometrical construction. For example:—It is required to bisect a given straight line.

61. In a proposition we usually have the General Enunciation, The Particular Enunciation, the Construction and Proof.

(*1*) The **General Enunciation** is a preliminary statement describing in general terms the geometrical construction to be made, or the geometrical truth to be established.

(*2*) The **Particular Enunciation** states the General Enunciation in special terms and refers it to a particular figure.

(*3*) The **Construction** shows what straight lines and circles should be drawn to effect the purpose of the problem or to prove the truth of the theorem.

(*4*) The **Proof** shows that the problem has been solved or that the property of the theorem is true.

In the enunciation of a theorem the **Hypothesis** is that which is assumed to be true, and the **Conclusion** is that which has to be proved.

For example in Art. 36 "If two straight lines cut each other the vertically opposite angles are equal to each other," the **hypothesis** is what is assumed to be true, viz., that two straight lines cut each other. The **Conclusion** is that which has to be proved, viz., "The vertically opposite angles are equal."

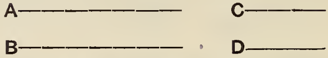
62. The demonstration of a theorem depends either on definitions and axioms, or on other theorems that have been previously shown to be true.

The following are some of the axioms commonly used in geometrical reasoning:—

1. Things that are equal to the same thing are equal to each other.

If $A = B$, $B = C$, $C = D$, $D = E$ and $E = F$, what about A and F?

2. If equals be added to equals the sums are equal.



Thus if A, B, C, D be four st. lines such that A = B and C = D, then the sum of A and C = the sum of B and D.

Exercise:—Mark four successive points A, B, C, D on a st. line such that AB = CD. Show that AC = BD.

3. If equals be taken from equals the remainders are equal.

Give example.

Exercise:—Mark four successive points A, B, C, D on a st. line such that AC = BD. Show that AB = CD.

4. If equals are added to unequals the sums are unequal, the greater sum being obtained from the greater unequal.

Give example. Show also, by example, that if unequals are added to unequals the sums may be either equal or unequal.

5. If equals are taken from unequals the remainders are unequal, the greater remainder being obtained from the greater unequal.

6. Doubles of the same thing, or of equal things, are equal to each other.

7. Halves of the same thing, or of equal things, are equal to each other.

8. The whole is greater than its part, and equal to the sum of all its parts.

Give examples.

9. Magnitudes that coincide with each other, are equal to each other.

These simple propositions, and others that are also plainly true, may be freely used in proving theorems.

BOOK I.

TRIANGLES

CONGRUENCE OF TRIANGLES

CONSTRUCTIONS

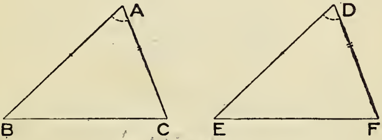
INEQUALITIES

First Case of the Congruence of Triangles

Proposition 1 Theorem

General enunciation.

If two triangles have two sides and the contained angle of one respectively equal to two sides and the contained angle of the other, the two triangles are congruent.



particular enunciation.

Hypothesis.—ABC and DEF are two △s having AB = DE, AC = DF and ∠ A = ∠ D.

Prove that (1) BC = EF,

(2) ∠ B = ∠ E,

(3) ∠ C = ∠ F,

(4) area of △ ABC = area of △ DEF; and, hence, △ ABC ≡ △ DEF.

 Congruent.

Proof.—Let $\triangle$ ABC be applied to $\triangle$ DEF so that vertex A falls on vertex D and AB falls along DE.

$\because$ AB = DE,

$\therefore$ vertex B must fall on vertex E.

$\because$ $\angle$ A = $\angle$ D,

$\therefore$ AC must fall along DF,

and $\therefore$, as AC = DF,

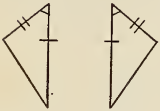
the vertex C must fall on the vertex F.

$\therefore$ $\triangle$ ABC coincides with $\triangle$ DEF.

Then BC = EF, $\angle$ B = $\angle$ E, $\angle$ C = $\angle$ F, and the area of $\triangle$ ABC = the area of $\triangle$ DEF,

and $\therefore$ $\triangle$ ABC $\equiv$ $\triangle$ DEF.

63.—Exercises



1. Prove Proposition 1 when one $\triangle$ has to be supposed to be turned over before it can be made to coincide with the other.

2. The $\angle$ B of a $\triangle$ ABC is a rt. $\angle$, and CB is produced to D making BD = BC. Prove AD = AC.

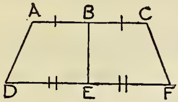
3. A, B, C are three points in a st. line such that AB = BC. DB is $\perp$ AC. Show that any point in DB, produced in either direction, is equidistant from A and C.

4. Two st. lines AOB, COD cut one another at O, so that OA = OB and OC = OD; join AD and BC, and prove $\triangle$s AOD, BOC congruent.

5. Prove that all chords of a circle which subtend equal angles at the centre are equal to one another.

6. If with the same centre O, two circles be drawn, and st. lines ODB, OEC be drawn to meet the circumferences in D, E, B, C; prove that BE = DC.

7. **ABCD** is a quadrilateral having the opposite sides **AB**, **CD** equal and ∠ **B** = ∠ **C**. Show that **AC** = **BD**.



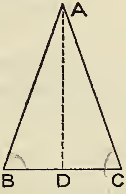
8. In the diagram, **ABC** and **DEF** are both ⊥ **BE**. Also **AB** = **BC** and **DE** = **EF**. Prove that **AD** = **CF**.

9. Two st. lines **AOB**, **COD** cut one another at rt. ∠s at **O**. **AO** is cut off = **OB**, and **CO** = **OD**. Prove that the quadrilateral **ACBD** is a rhombus.

10. Two quadrilaterals **ABCD**, **EFGH** have **AB** = **EF**, **BC** = **FG**, **CD** = **GH**, ∠ **B** = ∠ **F**, ∠ **C** = ∠ **G**. Prove that they are congruent.

Proposition 2 Theorem

The angles at the base of an isosceles triangle are equal to each other.



Hypothesis.—**ABC** is an isosceles △ having **AB** = **AC**.

Prove that ∠ **B** = ∠ **C**.

Hypothetical Construction.—Draw the st. line **AD** to represent the bisector of ∠ **BAC**.

Proof.—In the two △s **ADB**, **ADC**,

$$\left\{\begin{array}{lr} AB = AC, & (\textit{Hyp.}) \\ AD \text{ is common}, & \\ \angle BAD = \angle CAD, & (\textit{Const.}) \end{array}\right.$$

∴ △ **ADB** ≡ △ **ADC**, (I—1)

∴ ∠ **B** = ∠ **C**.

64.—Exercises

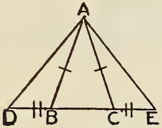
1. An equilateral △ is equiangular. Find, with the protractor, the number of degrees in the equal ∠s.

2. Make an isosceles △ with base 3 cm. and each of the sides 5 cm. Measure the base ∠s. (Ans. $72\frac{1}{2}°$.)

3. **ABC** is an equilateral △, and points **D, E, F,** are taken in **BC, CA, AB** respectively, such that **BD** = **CE** = **AF.** Show that **DEF** is an equilateral △.

4. Show that the exterior ∠s at the base of an isosceles △ are equal to each other.

5. The opposite ∠s of a rhombus are equal to each other.



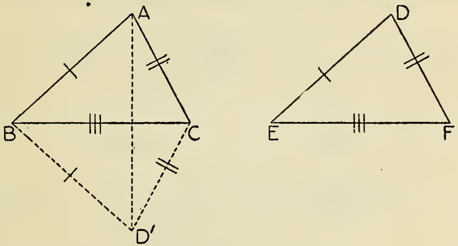
6. **ABC** is an isosceles △ having **AB** = **AC,** and the base **BC** produced to **D** and **E** such that **BD** = **CE.** Prove that **ADE** is an isosceles △.

7. **AC, AD** are two st. lines on opposite sides of **AB.** Prove that if the bisectors of ∠s **BAC, BAD** are at rt. ∠s, **AC, AD** must be in the same st. line.

Second Case of the Congruence of Triangles

Proposition 3 Theorem

If two triangles have the three sides of one respectively equal to the three sides of the other, the two triangles are congruent.



Hypothesis.—ABC, DEF are two $\triangle$s having AB = DE, AC = DF and BC = EF.

Prove that $\triangle$ ABC $\equiv$ $\triangle$ DEF.

Proof.—Let $\triangle$ DEF be applied to $\triangle$ ABC so that the vertex E falls on the vertex B and EF falls along BC.

Then $\because$ EF = BC, the vertex F falls on C. Let D take the position D′ on the side of BC remote from A.

Join AD′.

$$\because \qquad BA = BD',$$

$$\therefore \qquad \angle BAD' = \angle BD'A. \qquad \text{(I—2)}$$

$$\text{Similarly} \qquad \angle CAD' = \angle CD'A.$$

$$\therefore \angle BAD' + \angle CAD' = \angle BD'A + \angle CD'A,$$

$$i.e., \qquad \angle BAC = \angle BD'C = \angle EDF.$$

$$\text{Then in } \triangle\text{s BAC, EDF} \left\{ \begin{array}{r} BA = ED \\ CA = FD \\ \angle BAC = \angle EDF \end{array} \right.$$

$$i.e., \qquad \triangle ABC \equiv \triangle DEF. \qquad \text{(I—1)}$$

Note.—In the proof of this theorem three cases may occur:—AD′ may cut BC as in Fig. 1, or not cut BC as in Fig. 2, or pass through one end of BC as in Fig. 3.

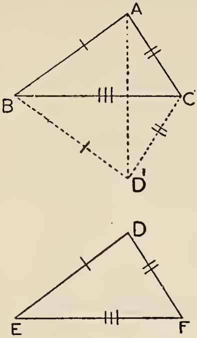


Fig. 1

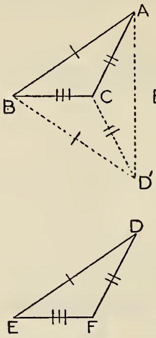


Fig. 2

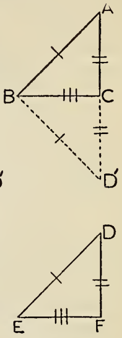


Fig. 3

The proof given above is that of the first case. The pupil should work out the proofs of the other two cases.

65.—Exercises

1. If the opposite sides of a quadrilateral be equal, the opposite $\angle$s are equal.

2. A diagonal of a rhombus bisects each of the $\angle$s through which it passes.

3. If in a quadrilateral ABCD the sides AB, CD be equal and $\angle$ ABC = $\angle$ BCD, prove that $\angle$ CDA = $\angle$ DAB.

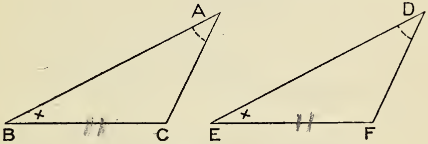
4. Show that equal chords in a circle subtend equal $\angle$s at the centre.

5. Prove that the diagonals of a rhombus bisect each other at rt. $\angle$s.

Third Case of the Congruence of Triangles

Proposition 4 Theorem

If two triangles have two angles and a side of one respectively equal to two angles and the corresponding side of the other, the triangles are congruent.



Hypothesis.—ABC, DEF are two $\triangle$s having $\angle A = \angle D$, $\angle B = \angle E$, and $BC = EF$.

Prove that $\triangle ABC \equiv \triangle DEF$.

Proof.— $\because \angle A = \angle D$,

and $\angle B = \angle E$,

$\therefore \angle A + \angle B = \angle D + \angle E$.

But $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$.

$\therefore \angle C = \angle F$.

Apply $\triangle ABC$ to $\triangle DEF$ so that BC coincides with the equal side EF.

$\because \angle B = \angle E$,

$\therefore$ BA falls along ED, and A is on the line ED.

$\because \angle C = \angle F$,

$\therefore$ CA falls along FD, and A is on the line FD.

But D is the only point common to ED and FD,

$\therefore$ A falls on D.

$\therefore \triangle ABC$ coincides with $\triangle DEF$,

and $\therefore \triangle ABC \equiv \triangle DEF$.

66.—Exercises

1. **ABC** is an equilateral triangle. The angle **A** is bisected by **AD** which meets the side **BC** in **D**. Prove △ **ABD** ≡ △ **ACD**. What is the size of each angle at **D**?

2. If the bisector of an ∠ of a △ be perpendicular to the opposite side, the △ is isosceles.

3. If two circles cut at two points, the st. line which joins their centres bisects at rt. ∠s the st. line joining the points of section.

4. **ABC** is an isosceles △. **DBC** is another isosceles △ on the same base **BC** and on the same side of it. Prove **AD** produced bisects **BC** at rt. ∠s.

5. If in a △ the perpendicular from the vertex on the base, bisects the base, the △ is isosceles.

6. Each ∠ of an equilateral △ is an ∠ of 60°.

7. On **AB**, **AC**, sides of an equilateral △ **ABC**, equilateral △s **ABD**, **ACE** are described externally. Prove **DC** = **BE**.

67. In Theoretical Geometry the use of instruments in making constructions is generally restricted to an ungraduated straight edge and a pair of compasses. With these instruments we can:—

1. Draw a st. line from one point to another.

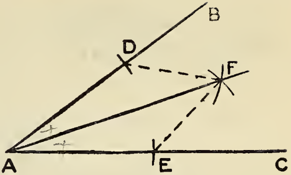
2. Produce a st. line.

3. Describe a circle with any point as its centre and radius equal to any given st. line.

4. Cut off from one st. line a part equal to another st. line.

Note.—*All constructions should be accurately and neatly drawn by the pupil, and, by means of theorems already proved, the correctness of the method of construction should be shown.*

Proposition 5 Problem



Bisect a given angle.

Let BAC be the given $\angle$.

Construction.—With the compasses cut off equal distances AD and AE from the arms of the $\angle$.

With centre D describe an arc.

With centre E and the same radius describe another arc cutting the first at F.

Join AF.

Then AF is the bisector of $\angle$ BAC.

Proof.—Join DF, EF.

$$\text{In } \triangle, \text{ADF, AEF,} \begin{cases} \text{AD} = \text{AE}, \\ \text{AF is common}, \\ \text{FD} = \text{EF}, \end{cases}$$

$\therefore \angle \text{DAF} = \angle \text{EAF}$. (I—3)

$\therefore$ AF bisects $\angle$ BAC.

Note.—*The equal radii for the arcs with centres* D *and* E *must be taken long enough for the arcs to intersect.*

68.—Exercises

1. Divide a given $\angle$ into four equal parts.

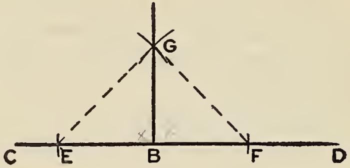
2. Prove that the bisectors of a pair of vertically opposite $\angle$s are in the same st. line.

3. Bisect a st. $\angle$.

4. Make a $\triangle$ having sides 42, 56, and 63 mm. Bisect the $\angle$ opposite the greatest side, produce the bisector to cut the side, and measure the segments. (Ans., 27 and 36 mm.)

Proposition 6 Problem

Draw a perpendicular to a given straight line from a given point in the line.



Let CD be the given st. line and B the given point.

Construction.—With centre B and any radius make BE = BF.

With centre E and any radius greater than EB make an arc. With the same radius and centre F make an arc cutting the other arc at G.

Draw GB.

Then GB is $\perp$ CD.

Proof.—Draw EG, FG.

$$\text{In } \triangle\text{s GBE, GBF, } \begin{cases} \text{EB} = \text{BF}, \\ \text{BG is common}, \\ \text{GE} = \text{GF}, \end{cases}$$

$$\therefore \angle \text{GBE} = \angle \text{GBF}. \qquad \text{(I—3)}$$

But these are supplementary $\angle$s

$\therefore$ each is a rt. $\angle$;

$\therefore$ GB $\perp$ CD.

69. The **altitude** of a triangle is the length of the perpendicular from any vertex to the opposite side.

70.—Exercises

1. Using ruler and compasses only, construct $\angle$s of (1), $45°$; (2), $22\frac{1}{2}°$; (3), $135°$; (4), $67\frac{1}{2}°$; (5), $225°$. reflex

2. Construct a rt.-$\angle$d $\triangle$ having one of the arms of the rt. $\angle$ three times the other.

3. Construct a rt.-$\angle$d $\triangle$ having the hypotenuse three times one of the arms of the rt. $\angle$.

4. Given the length of the hypotenuse and of one of the sides of a rt.-$\angle$d $\triangle$, construct the $\triangle$.

5. Construct a rhombus having each of its diagonals equal to twice a given st. line.

6. Construct a rhombus having one diagonal twice and the other four times a given st. line.

7. Construct an isosceles $\triangle$ having given its altitude and the length of one of the equal sides.

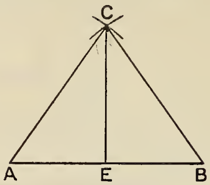
8. Construct an isosceles rt.-$\angle$d $\triangle$.

71. **Definitions.**—Sometimes when a proposition has been proved the truth of another proposition follows as an immediate consequence of the former; such a proposition is called a **corollary.**

A straight line which bisects a line of given length at right angles is called the **right bisector** of the line.

Proposition 7 Problem

Bisect a given straight line.



Let AB be the given st. line.

Construction.—With centre A and any radius that is plainly greater than half of AB, draw an arc.

With centre B and the same radius draw another arc cutting the first arc at C.

Join AC, CB.

Bisect $\angle$ ACB by CE meeting AB in E.

E is the middle point of AB.

Proof.—

$$\text{In } \triangle\text{s ACE, BCE, } \begin{cases} \text{AC} = \text{BC}, \\ \text{CE is common}, \\ \angle\ \text{ACE} = \angle\ \text{BCE}. \end{cases}$$

$\triangle$s ACE and BCE are congruent.

$\therefore$ AE = EB. (I—1)

Corollary.—From the above proof it follows that **the $\angle$s at E are rt. $\angle$s,** and hence, **CE is the right bisector of AB.**

72. **Definition.**—The straight line drawn from a vertex of a triangle to the middle point of the opposite side is called a **median** of the triangle.

73.—Exercises

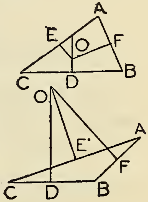
1. Divide a given st. line into four equal parts.

2. In an isosceles $\triangle$ prove that the bisector of the vertical $\angle$ is a median of the $\triangle$.

3. In an equilateral $\triangle$ prove that the bisectors of the $\angle$s are medians of the $\triangle$.

4. Show that any point in the right bisector of a given st. line is equidistant from the ends of the given line.

5. In any $\triangle$ the point of intersection of the right bisectors of any two sides is equidistant from the three vertices.

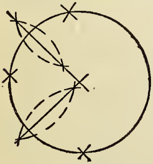


6. The right bisectors of the three sides of a $\triangle$ pass through one point.

The right bisectors of **AB, BC** *meet at* **O**. *Bisect* **AC** *at* **E**. *Join* **EO**. *Prove* **OE $\perp$ AC.**

7. Describe a circle through the three vertices of a $\triangle$.

8. Describe a circle to pass through three given points that are not in the same st. line.



9. Show how any number of circles may be drawn through two given points.

What line contains the centres of all these circles?

10. In a given st. line find a point that is equally distant from two given points.

11. On a given base describe an isosceles $\triangle$ so that the sum of the two equal sides may equal a given st. line.

In what case is this impossible?

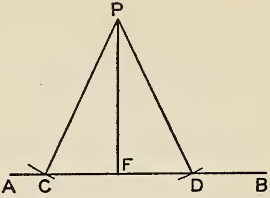
12. Construct a rhombus having its diagonals equal to 60 and 32 mm. Measure the side. (Ans. 34 mm.)

13. In $\triangle$ **ABC** find in **CA**, produced if necessary, a point **D** so that **DC = DB.**

14. In $\triangle$s **ABC, DEF, AB = DE, AC = DF** and the medians drawn from **B** and **E** are equal to each other. Prove that $\triangle$ **ABC** $\equiv$ $\triangle$ **DEF.**

PROPOSITION 8 PROBLEM

Draw a perpendicular to a given straight line from a given point without the line.



Let P be the given point and AB the given st. line.

Construction.—Describe an arc with centre P to cut AB at C and D.

Join PC, PD.

Bisect ∠ CPD by PF, which cuts AB at F.

PF is the required perpendicular.

Proof.—

In △s CPF, DPF, $\begin{cases} \text{CP} = \text{DP}, \\ \text{PF is common}, \\ \angle\ \text{CPF} = \angle\ \text{DPF}. \end{cases}$

∴ ∠ PFC = ∠ PFD. (I—1.)

But these are supplementary ∠s

∴ each is a rt. ∠;

∴ PF ⊥ AB.

74.—Exercises

1. Draw a △ ABC having $a = 42$, $b = 45$, $c = 39$ mm. Draw the ⊥ from A to BC and measure its length. (Ans. 36 mm.)

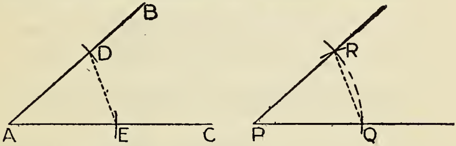
2. Draw a △ ABC having $a = 3$, $b = 5$, $c = 7$ cm. Find the length of the ⊥ from A to BC. (Ans. 43 mm. nearly.)

3. Draw a $\triangle$ ABC having $a = 11$, $b = 4$, $c = 9$ cm. Draw AX $\perp$ BC. Bisect BC at D and join AD. Bisect the $\angle$ A and let the bisector cut BC at H. Measure AX, AD and AH. (Ans. AX = 31 mm. nearly, AD = 43 mm. nearly, AH = 32 mm. nearly.)

4. Draw any irregular $\triangle$ and draw the $\perp$s from the three vertices to the opposite sides. If the drawing is correct the three vertices will pass through a common point.

Proposition 9 Problem

Construct an angle equal to a given angle.



Let BAC be the given $\angle$.

Construction.—From AC, AB cut off equal parts AE, AD.

Draw a line and mark a point P in it.

Cut off PQ = AE.

With centre P and radius PQ describe an arc.

With centre Q and radius DE describe an arc cutting the arc with centre P at R.

Join RP.

RPQ is the required $\angle$.

Proof.—Join DE, RQ.

In $\triangle$s PRQ, ADE, $\begin{cases} PQ = AE, \\ PR = AD, \\ RQ = DE, \end{cases}$

$\therefore$ $\triangle$s PRQ and ADE are congruent.

$\therefore$ $\angle$ RPQ = $\angle$ BAC. (I—3)

75.—Exercises

1. Construct a rhombus having given one of its $\angle$s and the length of one of its equal sides.

2. Make a $\triangle$ ABC, having $a = 30, b = 35$ and $c = 45$ mm. Make another $\triangle$, having one $\angle$ = A, and the two sides which contain this $\angle$ respectively 7 and 9 cm. Measure the third side. (Ans. 6 cm.)

3. Make a $\triangle$ ABC having $a = 4$, $b = 5$, $c = 6$ cm. Make another $\triangle$ A′B′C′ having B′ = B, C′ = C and $a' = 9$ cm. Measure b' and c'. (Ans. $b' = 11{\cdot}25$ cm. $c' = 13{\cdot}5$ cm.)

4. Construct an $\angle$ equal to the complement of a given acute $\angle$.

5. Construct an $\angle$ equal to the supplement of a given $\angle$.

6. On a given base describe an isosceles $\triangle$ having its altitude equal to a given st. line.

7. In the side BC of a $\triangle$ ABC find a point E, such that AE is half the sum of AB and AC.

8. The $\triangle$ formed by joining the middle points of the three sides of an isosceles $\triangle$ is isosceles.

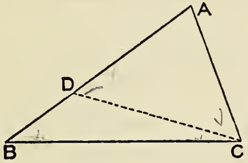
9. AB is a given st. line and C is a given point without the line. Find the point D so that AB is the right bisector of CD.

10. C, D are given points, (1) on opposite sides, (2) on the same side of a given st. line AB. Find a point P in AB so that CP, DP make equal $\angle$s with AB.

11. Construct a quadrilateral equal in all respects to a given quadrilateral.

PROPOSITION 10 THEOREM

If one side of a triangle be greater than another side, the angle opposite the greater side is greater than the angle opposite the less side.



Hypothesis.—ABC is a $\triangle$ having $AB > AC$.

Prove that $\angle ACB > \angle ABC$.

Construction.—From AB cut off $AD = AC$. Join DC.

Proof.—In $\triangle$ ADC,

$\because$ $AD = AC$,

$\therefore$ $\angle ADC = \angle ACD$. (I—2.)

But $\angle ACB > \angle ACD$,

$\therefore$ $\angle ACB > \angle ADC$.

In $\triangle$ BDC,

$\because$ BD is produced to A,

$\therefore$ exterior $\angle ADC >$ interior and non-adjacent $\angle DBC$.

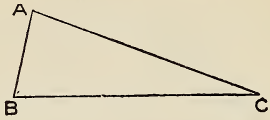
But $\angle ACB > \angle ADC$;

much more $\therefore$ is $\angle ACB > \angle ABC$.

PROPOSITION 11 THEOREM

(*Converse of I—10*)

If one angle of a triangle be greater than another angle of the same triangle, the side opposite the greater angle is greater than the side opposite the less.



Hypothesis.—In $\triangle$ ABC $\angle$ B $>$ $\angle$ C.

Show that AC $>$ AB.

Proof.—If AC be not $>$ AB,

then either AC $=$ AB,

or AC $<$ AB.

If AC $=$ AB,

then $\angle$ B $=$ $\angle$ C. (I—2.)

But this is not so, $\therefore$ AC is not $=$ AB.

If AC $<$ AB,

then $\angle$ B $<$ $\angle$ C. (I—10.)

But this also is not so, $\therefore$ AC is not $<$ AB.

Hence $\because$ AC is neither $=$ nor $<$ AB,

$\therefore$ AC $>$ AB.

76. This method of proof, in which we begin by assuming that the conclusion of the theorem is not true, and then show that something absurd or contrary to the hypothesis must follow, is called the **indirect method of demonstration.**

77. Compare propositions (10) and (11).

In proposition (10) the hypothesis is that one side of a triangle is greater than another side; the conclusion is that the angle opposite the greater side is greater than the angle opposite the less side.

In proposition (11) the hypothesis is that one angle of a triangle is greater than another angle of the same triangle; the conclusion is that the side opposite the greater angle is greater than the side opposite the less.

Thus in these propositions the hypothesis of each is the conclusion of the other.

When two propositions are such that the hypothesis of each is the conclusion of the other, they are said to be **converse propositions**; or each is said to be the converse of the other.

The converse of a true proposition may, or may not, be true. The converse propositions in Theorems 10 and 11 are both true; but consider the true proposition:— All rt. $\angle$s are equal to one another; and its converse:— All equal $\angle$s are rt. $\angle$s. The last is easily seen to be untrue. Consequently, proof must in general be given for each of a pair of converse propositions.

When a proposition is known to be true and we wish to prove the converse we commonly use the indirect method.

78.—Exercises

1. The perpendicular is the shortest st. line that can be drawn from a given point to a given straight line.

The length of the $\perp$ from a given point to a given st. line is called the distance of the point from the line.

2. **ABCD** is a quadrilateral, of which **AD** is the longest side, and **BC** the shortest. Show that $\angle$ **B** > $\angle$ **D**, and that $\angle$ **C** > $\angle$ **A**.

3. The hypotenuse of a rt.-$\angle$d $\triangle$ is greater than either of the other two sides.

4. A st. line drawn from the vertex of an isosceles $\triangle$ to any point in the base is less than either of the equal sides.

5. A st. line drawn from the vertex of an isosceles $\triangle$ to any point in the base produced is greater than either of the equal sides.

6. If one side of a $\triangle$ be less than another, the $\angle$ opposite the less side is acute.

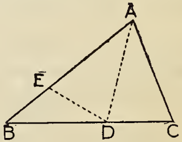
7. If **D** be any point in the side **BC** of a $\triangle$ **ABC**, the greater of the sides **AB**, **AC**, is greater than **AD**.

8. **AB** is drawn from **A** $\perp$ **CD**. **E**, **F** are two points in **CD** on the same side of **B**, and such that **BE** < **BF**. Show that **AE** < **AF**. Prove the same proposition when **E**, **F** are on opposite sides of **B**.

9. **ABC** is a $\triangle$ having **AB** > **AC**. The bisector of $\angle$ **A** meets **BC** at **D**. Show that **BD** > **DC**. Give a general statement of this proposition.

10. **ABC** is a $\triangle$ having **AB** > **AC**. If the bisectors of $\angle$s **B**, **C** meet at **D**, show that **BD** > **DC**.

11. Prove Theorem 10 from the following construction: Bisect $\angle$ **A** by **AD** which meets **BC** at **D**; from **AB** cut off **AE** = **AC**, and join **ED**.



12. The $\angle$s at the ends of the greatest side of a $\triangle$ are acute.

13. If **AB** > **AD** in the ||gm **ABCD**, $\angle$ **ADB** > $\angle$ **BDC**.

PROPOSITION 12 THEOREM

(Converse of I—2)

If two angles of a triangle are equal to each other, the sides opposite these equal angles are equal to each other.

Hypothesis.—In $\triangle$ ABC $\angle$ B $=$ $\angle$ C.

Prove that AB $=$ AC.

Proof.— If AB is not $=$ AC,

let AB $>$ AC.

Then $\angle$ C $>$ $\angle$ B. (I—10.)

But this is not so.

$\therefore$ AB is not $>$ AC.

Similarly it may be shown that

AB is not $<$ AC.

$\therefore$ AB $=$ AC.

79.—Exercises

1. An equiangular △ is equilateral.

2. **BD, CD** bisect the ∠s **ABC, ACB** at the base of an isosceles △ **ABC.** Show that △ **DBC** is isosceles.

3. **ABC** is a △ having **AB, AC** produced to **D, E** respectively. The exterior ∠s **DBC, ECB** are bisected by **BF, CF,** which meet at **F.** Show that, if **FB = FC,** the △ **ABC** is isosceles.

4. On the same side of **AB** the two △s **ACB, ADB** have **AC = BD, AD = BC,** and **AD, BC** meet at **E.** Show that **AE = BE.**

5. On a given base construct a △ having one of the ∠s at the base equal to a given ∠, and the sum of the sides equal to a given st. line.

6. On a given base construct a △ having one of the ∠s at the base equal to a given ∠ and the difference of the sides equal to a given st. line.

7. A st. line drawn ⊥ to **BC,** the base of an isosceles △ **ABC,** cuts **AB** at **X** and **CA** produced at **Y.** Show that **AXY** is an isosceles △.

8. The middle point of the hypotenuse of a rt.-∠d △ is equidistant from the three vertices.

9. Construct a rt.-∠d △, having the hypotenuse equal to one given st. line, and the sum of the other two sides equal to another given st. line.

10. If one ∠ of a △ equals the sum of the other two, show that the △ is a rt.-∠d △.

The Ambiguous Case in the Comparison of Triangles

Proposition 13 Theorem

If two triangles have two sides of one respectively equal to two sides of the other and have the angles opposite one pair of equal sides equal to each other, the angles opposite the other pair of equal sides are either equal or supplementary.

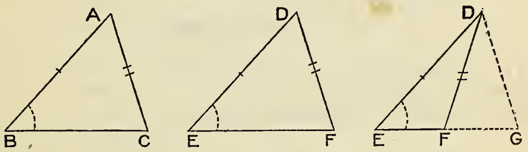


Fig. 1 Fig. 2

Hypothesis.—ABC, DEF are two $\triangle$s having AB = DE, AC = DF and $\angle$ B = $\angle$ E.

Prove that either $\angle$ C = $\angle$ F,

or $\angle$ C + $\angle$ F = two rt. $\angle$s.

Proof.—The $\angle$s A and D are either equal or unequal.

Case I. Suppose $\angle$ A = $\angle$ D. (Fig. 1.)

Then in the two $\triangle$s ABC, DEF,

$\because$ $\angle$ A = $\angle$ D,

$$\text{and } \angle B = \angle E,$$

$$\therefore \angle A + \angle B = \angle D + \angle E.$$

$$\text{But } \angle A + \angle B + \angle C = \angle D + \angle E + \angle F.$$

$$\therefore \angle C = \angle F.$$

Case II. Suppose $\angle A$ not $= \angle D$. (Fig. 2)

Make $\angle EDG = \angle BAC$, and produce its arm to meet EF, produced if necessary, at G.

$$\text{In } \triangle\text{s } ABC, DEG, \left\{ \begin{array}{l} \angle A = \angle EDG, \\ \angle B = \angle E, \\ AB = DE, \end{array} \right.$$

$$\left. \begin{array}{l} \therefore \angle C = \angle G, \\ \text{and } AC = DG. \end{array} \right\} \qquad \text{(I—4.)}$$

$$\text{But } DF = AC, \qquad (\textit{Hyp.})$$

$$\therefore DF = DG.$$

$$\therefore \angle DFG = \angle G. \qquad \text{(I—2.)}$$

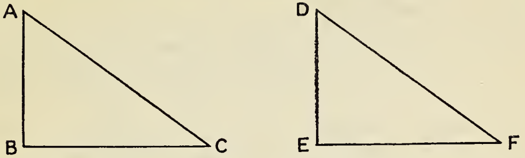
$$\text{But } \angle C = \angle G.$$

$$\therefore \angle C = \angle DFG.$$

$$\angle DFG + \angle DFE = \text{two rt. } \angle\text{s},$$

$$\therefore \angle C + \angle DFE = \text{two rt. } \angle\text{s}.$$

Cor.—If two right-angled triangles have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the triangles are congruent.



Hypothesis.—ABC, DEF are two triangles having right angles at B and E and AB = DE, AC = DF.

Prove that the $\triangle$s ABC, DEF are congruent.

Proof.—By proposition 13 the angles at C and F are either equal or supplementary.

As the angles at B and E are right angles the angles at C and F must be acute angles and therefore cannot be supplementary.

Hence $\angle C = \angle F$.

$$\text{In } \triangle\text{s ABC, DEF, } \begin{cases} \angle B = \angle E, \\ \angle C = \angle F, \\ AC = DF. \end{cases}$$

$\therefore \triangle ABC \equiv \triangle DEF.$ (I—4.)

80. There are **six** parts in a triangle, viz., **three sides** and **three angles,** and in the cases in which the **congruence** of two triangles has been established three parts of one triangle, one at least a side, have been given respectively equal to the corresponding parts of the other.

The following general cases occur:—

1. Two sides and the contained angle. The triangles are congruent—Proposition 1.

2. Three sides. The triangles are congruent—Proposition 3.

3. Two angles and a side. The triangles are congruent—Proposition 4.

4. Two sides and an angle opposite one of them. In this case the triangles are congruent if the angle is opposite the greater of the two sides—§81, Ex. 2, but if the angle is opposite the shorter of the two sides, they are not necessarily congruent—Proposition 13.

5. Three angles. The triangles are not necessarily congruent—§81, Ex. 6.

81.—Exercises

1. If the bisector of the vertical $\angle$ of a $\triangle$ also bisects the base the $\triangle$ is isosceles.

2. If two $\triangle$s have two sides of one respectively equal to two sides of the other and the $\angle$s opposite the greater pair of equal sides equal to each other, the $\triangle$s are congruent.

3. Construct a $\triangle$ having given two sides and the $\angle$ opposite one of them.

When will there be: (*a*) no solution, (*b*) two solutions, (*c*) only one solution?

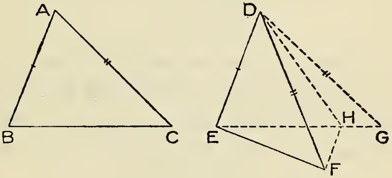
4. If two $\angle$s of a $\triangle$ are bisected and the bisectors are produced to meet, the line joining the point of intersection to the vertex of the third $\angle$ bisects that third $\angle$. Hence.—**The bisectors of the three $\angle$s of a $\triangle$ pass through one point.**

5. If two exterior $\angle$s of a $\triangle$ are bisected and the bisectors are produced to meet, the line joining the point of intersection of the bisectors to the vertex of the third $\angle$ of the $\triangle$ bisects that third $\angle$.

6. Draw diagrams to show that if the three $\angle$s of one $\triangle$ are respectively equal to the three $\angle$s of another $\triangle$, the two $\triangle$s are not necessarily congruent.

PROPOSITION 14 THEOREM

If two triangles have two sides of one respectively equal to two sides of the other but the contained angle in one greater than the contained angle in the other, the base of the triangle which has the greater angle is greater than the base of the other.



Hypothesis.—ABC, DEF are two $\triangle$s having AB = DE, AC = DF and $\angle$ BAC > $\angle$ EDF.

Prove that BC > EF.

Construction.—Make $\angle$ EDG = $\angle$ BAC and cut off DG = AC, or DF. Join EG. Bisect $\angle$ FDG and let the bisector meet EG at H. Join FH.

Proof.—

$$\text{In } \triangle\text{s ABC, DEG, } \begin{cases} \text{AB} = \text{DE}, \\ \text{AC} = \text{DG}, \\ \angle \text{A} = \angle \text{EDG}. \end{cases}$$

$$\therefore \text{BC} = \text{EG}. \qquad \text{(I—1.)}$$

$$\text{In } \triangle\text{s FDH, GDH, } \begin{cases} \text{DF} = \text{DG}, \\ \text{DH is common}, \\ \angle \text{FDH} = \angle \text{GDH}, \end{cases}$$

$$\therefore \text{FH} = \text{HG}. \qquad \text{(I—1.)}$$

In $\triangle$ EHF, EH + HF > EF.

But HF = HG,

$\therefore$ EH + HG > EF.

i.e., EG > EF.

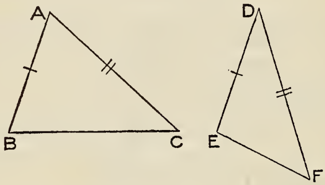
But BC = EG,

$\therefore$ BC > EF.

PROPOSITION 15 THEOREM

(Converse of I—14)

If two triangles have two sides of one respectively equal to two sides of the other but the base of one greater than the base of the other, the triangle which has the greater base has the greater vertical angle.



Hypothesis.—ABC, DEF are two $\triangle$s having AB = DE, AC = DF and BC > EF.

Prove that $\angle$ A > $\angle$ D.

Proof.— If $\angle$ A is not > $\angle$ D,

either $\angle$ A = $\angle$ D,

or $\angle$ A < $\angle$ D.

(1) If $\angle$ A = $\angle$ D.

In $\triangle$s ABC, DEF, $\left\{\begin{array}{l} \text{AB} = \text{DE}, \\ \text{AC} = \text{DF}, \\ \angle \text{A} = \angle \text{D}, \end{array}\right.$

$\therefore$ BC = EF. (I—1.)

But this is not so.

$\therefore \angle$ A is not $= \angle$ D.

(2) If $\angle A < \angle D$.

In $\triangle$s ABC, DEF, $\left\{ \begin{array}{l} AB = DE, \\ AC = DF, \\ \angle A < \angle D, \end{array} \right.$

$\therefore BC < EF$. (I—14.)

But this is not so.

$\therefore \angle$ A is not $< \angle$ D.

Then since $\angle$ A is neither $=$ nor $< \angle$ D,

$\therefore \angle A > \angle D$.

82.—Exercises

1. **ABCD** is a quadrilateral having **AB** = **CD** and $\angle$ **BAD** > $\angle$ **ADC**. Show that $\angle$ **BCD** > $\angle$ **ABC**.

2. In $\triangle$ **ABC, AB** > **AC** and **D** is the middle point of **BC**. If any point **P** in the median **AD** be joined to **B** and **C**, **BP** > **CP**.

If **AD** be produced to any point **Q** show that **BQ** < **QC**.

3. **D** is a point in the side **AB** of the $\triangle$ **ABC**. **AC** is produced to **E** making **CE** = **BD**. **BE** and **CD** are joined. Show that **BE** > **CD**.

4. If two chords of a circle are unequal, the greater subtends the greater angle at the centre.

5. Two circles have a common centre at **O**. **A, B** are two points on the inner circumference and **C, D** two on the outer. $\angle$ **AOC** > $\angle$ **BOD**. Show that **AC** > **BD**.

6. **CD** bisects **AB** at rt. $\angle$s. A point **E** is taken not in **CD**. Prove that **EA, EB** are unequal.

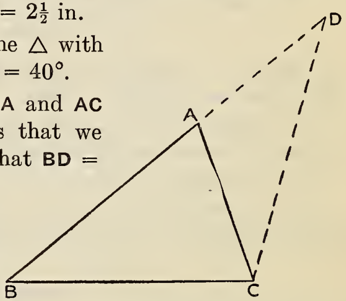
7. In $\triangle$ **ABC, AB** > **AC**. Equal distances **BD, CE** are cut off from **BA, CA** respectively. Prove **BE** > **CD**.

8. In $\triangle$ **ABC, AB** > **AC**. **AB, AC** are produced to **D, E** making **BD** = **CE**. Prove **CD** > **BE**.

Analysis of a Problem

83. A common method of discovering the solution of a problem begins with the drawing of the given figure or figures. The required part is then sketched in, and a careful examination is made to determine the connection between the given parts and the required result. Properties of the figure are noted, and lines are drawn that may help in finding the solution. This method of attack is known as the **Analysis of the Problem.** Its use is illustrated in the following examples:—

1. It is required to make a △ ABC having ∠ B = 40°, $a = 1\frac{1}{2}$ in. and $b + c = 2\frac{1}{2}$ in.



Make a sketch of the △ with BC = $1\frac{1}{2}$ in. and ∠ B = 40°.

Since the sum of BA and AC is given this suggests that we produce BA to D so that BD = $2\frac{1}{2}$ in.

It is now seen that, as BC, BD and ∠ B are known, the △ DBC can be drawn.

To complete the construction we must know where to draw the line AC. This difficulty is overcome when we notice that, since AD = AC, ∠ D = ∠ ACD.

The following construction for the problem is then evident:—

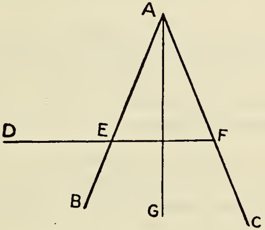
Make BC = $1\frac{1}{2}$ in., ∠ B = 40° and cut off BD = $2\frac{1}{2}$ in. Join DC, and make ∠ ACD = ∠ D.

ABC is the required △ and the proof is evident.

2. Construct a △ in which ∠ **B** = 38°, a = 46 mm., and $c - b$ = 24 mm.

3. Construct a △ with its perimeter = 11 cm. and its base ∠s 58° and 64°.

4. From **D** which is not in either of the lines **AB**, **AC**, draw a line **DEF** which shall cut off equal lengths **AE**, **AF** from **AB**, **AC**.



Suppose **DEF** drawn so that **AE** = **AF**, **E** and **F** being the undetermined points.

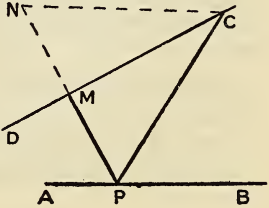
When the construction is made it will be necessary to prove **AE** = **AF**. This suggests an isosceles triangle. Also the line bisecting the vertical angle of an isosceles triangle is perpendicular to the base.

Hence, bisect ∠ **EAF** by **AG**.

Draw **DEF** ⊥ **AG** meeting **AC**, **AB** in **E** and **F**.

DEF is the required line.

5. Find a point **P** in a given st. line **AB** such that its distance from a given point **C** is double its distance from a given st. line **CD** through the point **C**.



Miscellaneous Exercises

1. A ladder, whose foot is placed 10 feet from the base of a house, reaches to a window 25 feet above the ground. Draw a plan in which 1″ represents 10 ft.; and find by measurement the length of the ladder.

2. By making a diagram (scale 1″ = 10 ft.) find the length of a shadow cast by a vertical flag pole 20 feet high when the sun is 50° above the horizon.

3. **B** and **C** are two points, known to be 350 yards apart, on a straight shore. **A** is a vessel at anchor. The angles **CBA**, **BCA** are observed to be 33° and 81° respectively. Find graphically the approximate distance of the vessel from the points **B** and **C**, and from the nearest point on shore.

4. Two objects **A** and **B** are separated by a body of water. In order to find the distance between them a third point **C** is chosen from which each of these points is visible and the following measurements are made: **CA** = 2,000 ft., **CB** = 3,000 ft. and **ACB** = 105°. By using a diagram find the distance from **A** to **B**.

5. It is required to find the distance between two points **A** and **B**; but as a lake intervenes, a direct measurement cannot be made. The surveyor therefore takes a third point **C**, from which both **A** and **B** are accessible, and he finds **CA** = 245 yards, **CB** = 320 yards, and the angle **ACB** = 42°. Ascertain from a plan the approximate distance between **A** and **B**.

6. There are two points **A**, **B** on the bank of a river but owing to a curve in its course it is impossible to measure the distance between them directly. A third point **C** is chosen such that the distances **AC** = 1,500 ft., and **BC** = 2,400 ft. can be measured, and the angle **ACB** is found to be 72°. By using a diagram find the distance from **A** to **B**.

(*The following deductions are to be done theoretically.*)

7. One angle of a triangle measures 53°. If the other two angles are equal, calculate their magnitude.

8. The three angles of a triangle are equal. Calculate the magnitude of each.

9. Each of the angles at the base of a triangle is double of the vertical angle. Calculate all three angles.

10. If one angle of a triangle is equal to the sum of the other two, what is the size of the first angle?

11. Divide a right-angled triangle into two isosceles triangles.

12. **ABC** is an isosceles triangle, having **AB** = **AC**. **F**, **E** are the middle points of **AB**, **AC** respectively. If **BE** and **CF** are joined, prove that **BE** = **CF** and ∠ **ABE** = ∠ **ACF**.

13. **ABC** is an isosceles triangle, having **AB** = **AC**. The angles at **B** and **C** are bisected by lines which meet at **D**. Prove that **DB** = **DC**.

14. **ABC** is an isosceles triangle having **AB** = **AC**, and **P** is any point on the line bisecting the angle **A**. Prove that **PBC** is an isosceles triangle.

15. **ABC** is an isosceles triangle, having **AB** = **AC**. The angles at **B** and **C** are bisected by lines which meet at **D**. Prove that **AD** bisects ∠ **BAC**.

16. **ABC** is a triangle, and the sides **AB** and **AC** are produced to **D** and **E**. If the angle **DBC** is equal to the angle **ECB**, prove that **AB** is equal to **AC**.

17. If **ABC**, **DBC** are two isosceles triangles drawn on opposite sides of the same base **BC**, and if **AD** be joined, prove that each of the angles **BAC**, **BDC** will be divided into two equal parts.

18. The equal sides **BA**, **CA** of an isosceles triangle **BAC** are produced beyond the vertex **A** to the points **E** and **F**, so that **AE** is equal to **AF**; and **FB**, **EC** are joined; show that **FB** is equal to **EC**.

19. Show that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal to one another.

20. Two given points in the base of an isosceles triangle are equidistant from the extremities of the base: show that they are also equidistant from the vertex.

21. **ABC** is an isosceles triangle. From the equal sides **AB**, **AC** two equal parts **AX**, **AY** are cut off, and **BY** and **CX** are joined. Prove that **BY** = **CX**.

22. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA; if DC is drawn, show that BCD is a right angle.

23. Show that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

24. Prove that the triangle formed by joining the three middle points of the three sides of an isosceles triangle is isosceles.

25. In an isosceles triangle either of the equal sides is greater than half the base.

26. An officer wishes to find his distance from an object due East of his position. A corporal crawls 52 yards North-East and finds himself due North of the object. What is the distance?

27. A straight line AOB is drawn on paper, which is then folded about O, so as to make OA fall along OB. Show that the crease left in the paper is perpendicular to AB.

28. Two straight lines AB, CD cross at O. If the angle BOD is bisected by OX, and AOC by OY, prove that OX, OY are in the same straight line.

29. If in the triangle ABC, the angles B and D be each double the angle A, and BD bisect the angle B, what three lines in the figure are equal to one another?

30. ABC is an isosceles triangle, and the equal angles at B and C are bisected by lines which meet in O. Show that BO = CO. Also show that AO bisects the angle at A.

31. Show that in the rhombus

(1) The opposite angles are equal.

(2) The diagonals bisect the angles through which they pass.

(3) The diagonals bisect one another at right angles.

32. If the bisectors of the angles B and C, of the triangle ABC, meet at O, show that the angle BOC $= 90° + \frac{1}{2}$ A.

33. If the bisectors of the exterior angles at B and C, of the triangle ABC, meet at O, show that the angle BOC $= 90° - \frac{1}{2}$ A.

34. Show that the bisector of the vertical angle of an isosceles triangle (1) bisects the base, (2) is perpendicular to the base.

35. **ACB**, **ADB** are two triangles on the same side of **AB**, with **AC** = **BD** and **AD** = **BC**. If **AD**, **BD** meet in **O**, prove that the triangles **OAB** and **OCD** are isosceles.

36. If the sides **AB**, **AC** of a triangle **ABC** be produced to **D** and **E**, and if the bisectors of the angles **BCE**, **CBD** meet in **O**, show that the perpendiculars from **O** on **BD**, **BC**, **CE** are all equal.

37. **FGH** is a triangle, having **FG** = **FH**. **GK** is drawn perpendicular to **FG** to meet **FH** produced at **K**, and **HL** is drawn perpendicular to **FH** to meet **FG** produced at **L**. Prove that **GL** = **HK**.

38. **PQR** is a triangle, having ∠ **PQR** = ∠ **PRQ**. **QS** is a line bisecting ∠ **PQR** and meeting **PR** at **S**; **RT** is a line bisecting ∠ **PRQ** and meeting **PQ** at **T**; **ST** is joined. Prove that ∠ **QTS** = ∠ **RST**.

39. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

40. Construct a triangle, having given the base, one of the angles at the base, and the difference of the sides.

41. From **D**, which is not in either of the lines **AB**, **AC**, draw a line **DEF** which shall cut off equal lengths **AE**, **AF** from **AB**, **AC**.

42. Construct a right-angled triangle, having given the lengths of the hypotenuse and of one side.

43. Construct an isosceles triangle, having given the vertical angle and the perpendicular from the vertical angle on the base.

44. Construct an isosceles triangle, having given its perimeter and the perpendicular from the vertex on the base.

45. **ABC** is a triangle, and the angle **A** is bisected by **AD**, meeting **BC** in **D**. Show that **BA** is greater than **BD**, and **CA** greater than **CD**.

46. The angles **ABC**, **ACB** of the triangle **ABC** are bisected by **OB**, **OC**. If **AB** be greater than **AC**, then **OB** is greater than **OC**.

47. In triangle **ABC**, side **AB** is greater than side **AC**. The angle **A** is bisected by a line meeting **BC** at **D**. Show that **BD** is greater than **CD**.

48. The difference of any two sides of a triangle is less than the third side.

49. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

50. The perimeter of a quadrilateral is greater than the sum of its diagonals.

51. In a triangle any two sides are together greater than twice the median which bisects the remaining side.

52. In any triangle the sum of the medians is less than the perimeter.

53. If **O** is any point within a triangle **ABC**, prove that angle **BOC** is greater than angle **BAC**. Also prove that **OB** + **OC** is less than **AB** + **AC**.

54. **ABC** is a triangle having the angle **ABC** greater than the angle **ACB**. If **AD** be drawn to the middle point of **BC**, show that the angle **ADC** is obtuse.

55. **ABC** is a triangle having **AB** less than **AC** and **P** is any point in the line joining **A** to the middle point of **BC**. Show that **P** is nearer to **B** than to **C**.

56. The side **AB** of the triangle **ABC** is greater than the side **AC**. From **BA**, **CA** equal parts **BD**, **CE** are cut off. Show that **BE** is greater than **CD**.

57. In a quadrilateral **ABCD**, given that **AD** = **BC**, **AC** = **BD**, prove that $\angle$ **DAB** = $\angle$ **CBA** and $\angle$ **ADC** = $\angle$ **BCD**.

58. In a quadrilateral **ABCD**, **AB** is the least and **CD** the greatest of the sides. Prove that $\angle$ **A** is greater than $\angle$ **C** and $\angle$ **B** is greater than $\angle$ **D**.

59. From two given points on the same side of a given line, draw two lines which shall meet in that line and make equal angles with it.

60. Through two given points on opposite sides of a given straight line, draw two straight lines which shall meet in the given straight line, and include an angle bisected by the given straight line.

61. **ABC** is an isosceles triangle; **DEF** is a straight line perpendicular to the base **BC**, meeting **AB** in **E** and **CA** produced in **F**; show that the triangle **AEF** is isosceles.

62. On the circumference of a circle whose centre is O, three points A, B, C are taken, such that the straight lines AB, BC are equal. Show that OB bisects AC at right angles; also that OB bisects the angles AOC, ABC.

63. The straight line joining the middle point of the hypotenuse of a right-angled triangle to vertex of the right angle is equal to half the hypotenuse.

64. If, in a right-angled triangle, one of the acute angles be double the other, show that the hypotenuse is double the smaller side. How many degrees are there in each angle of the figure?

65. If, in a right-angled triangle, the hypotenuse be double the smallest side, show that one of the acute angles is double the other.

66. In a triangle ABC, AD is perpendicular to BC, and AE bisects the angle at A. Show that the angle DAE is equal to half the difference between the angles at B and C.

67. TOK is a triangle with a right angle at O. The bisector of OTK meets OK in L. Prove that LK is greater than OL.

68. Construct a triangle giving two of its angles and the sum of the three sides.

69. ABC is a triangle, obtuse angled at C; straight lines are drawn bisecting CA, CB at rt. $\angle$s and cutting AB in D, E respectively. Prove $\angle$ DCE is equal to twice the excess of $\angle$ ACB over a rt. $\angle$.

70. In triangle ABC side BC is produced to D. Prove that the angle between the bisectors of angles ABC, ACD = half the angle A.

BOOK II.

PARALLELISM

PARALLEL LINES

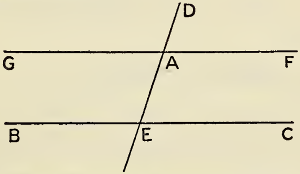
PARALLELOGRAMS AND TRIANGLES

AREAS

Parallel Lines

Proposition 1 Problem

Through a given point draw a straight line parallel to a given straight line.



Let A be the given point and BC the given straight line.

Construction.—Through A draw any line DAE cutting BC in E.

At A in the straight line DAE make ∠ DAF = ∠ AEC. (I—9.)

Produce FA to G.

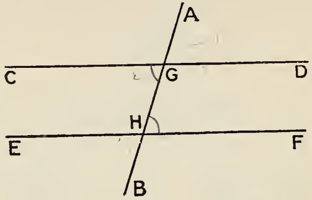
Then GF || BC.

Proof.—Because the straight line DAE falls on the straight lines GF, BC and makes the exterior ∠ DAF = interior and opposite ∠ AEC.

∴ GF drawn through A is || BC. (Int.—41.)

Proposition 2 Theorem

If a transversal cuts two parallel straight lines, the alternate angles are equal to each other.



Hypothesis.—The transversal AB cuts the || st. lines CD, EF at G, H.

Prove that ∠ CGH = ∠ GHF.

Proof.—Because AB falls across the two parallel straight lines CD and EF the ext. ∠ AGD = int. and opp. ∠ GHF. (Int. 41.)

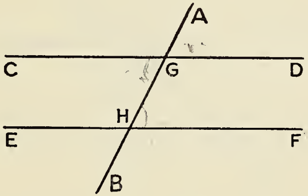
But ∠ AGD = ∠ CGH. (Int. 36)

∴ ∠ CGH = ∠ GHF.

Proposition 3 Theorem

(*Converse of II—2.*)

If a transversal meeting two straight lines makes the alternate angles equal to each other, the two straight lines are parallel.



Hypothesis.—The transversal AB cutting CD and EF makes $\angle$ CGH = $\angle$ GHF.

Prove that CD || EF.

Proof.— Because $\angle$ CGH = $\angle$ GHF, (*Hyp.*)

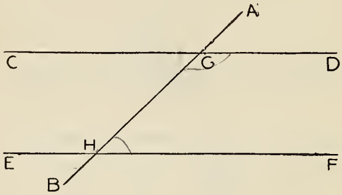
and $\angle$ CGH = $\angle$ AGD, (*Int.* 36.)

$\therefore$ $\angle$ AGD = $\angle$ GHF.

Hence CD is || EF. (Int 41),

Proposition 4 Theorem

If a transversal cuts two parallel straight lines, it makes the interior angles on the same side of the transversal supplementary.



Hypothesis.—AB cuts the || st. lines CD, EF.

Prove that $\angle$ DGH + $\angle$ GHF = two rt. $\angle$s.

Proof.— $\because$ $\angle$ GHF = $\angle$ CGH,

$\therefore$ $\angle$ GHF + $\angle$ DGH = $\angle$ CGH + $\angle$ DGH,

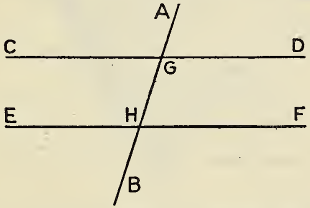
but $\angle$ CGH + $\angle$ DGH = a st. $\angle$.

$\therefore$ $\angle$s GHF, DGH are supplementary.

PROPOSITION 5 THEOREM

(*Converse of II—4*)

If a transversal cut two other straight lines and makes the interior angles on the same side of the transversal supplementary then the two lines are parallel.



Hypothesis.—AB cuts the straight lines CD, EF making $\angle$ DGH + $\angle$ GHF = two rt. $\angle$s.

Prove.—CD || EF.

Proof.— $\angle$ DGH + $\angle$ GHF = two rt. $\angle$s,

and $\angle$ DGH + $\angle$ AGD = two rt. $\angle$s.

$\therefore$ $\angle$ DGH + $\angle$ AGD = $\angle$ DGH + $\angle$ GHF.

Taking away the common $\angle$ DGH

$\angle$ AGD = $\angle$ GHF,

$\therefore$ CD || EF. (Int. 41)

84.—Exercises

1. Lines which are $\perp$ to the same st. line are || to each other.

2. If both pairs of opposite sides of a quadrilateral are equal to each other the quadrilateral is a ||gm.

3. A rhombus is a ||gm.

4. If one angle of a ||gm is a rt. ∠, the other three ∠s are also right angles.

5. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a ||gm.

6. If the diagonals of a quadrilateral bisect each other at rt. ∠s, the quadrilateral is a rhombus.

7. If the diagonals of a quadrilateral are equal and bisect each other, the quadrilateral is a rectangle.

8. If the diagonals of a quadrilateral are equal and bisect each other at rt. ∠s, the quadrilateral is a square.

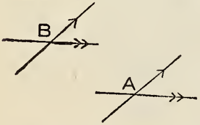
10. Prove by using a transversal that st. lines which are || to the same st. line are || to each other.

11. Through a given point draw a st. line making a given angle with a given st. line.

12. If through the vertex of an isosceles △ a line be drawn parallel to the base, it will bisect the exterior vertical angle.

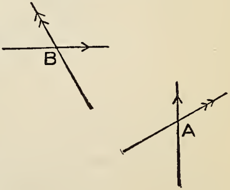
13. If the exterior vertical angle of isosceles △ be bisected, the bisector is parallel to the base.

14. If a st. line be terminated by two ||s, all st. lines drawn through its middle point and terminated by the same ||s are bisected at that point.



15. If two lines intersecting at **A** are respectively || to two lines intersecting at **B**, each ∠ at **A** is either equal to or supplementary to each ∠ at **B**.

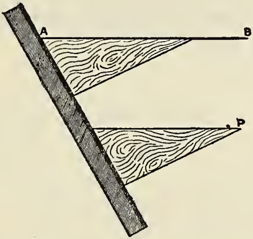
16. If two lines intersecting at **A** are respectively ⊥ to two lines intersecting at **B**, each ∠ at **A** is either equal to or supplementary to each ∠ at **B**.



17. If from any point in the bisector of an ∠ st. lines are drawn || to the arms of the ∠ and terminated by the arms, these st. lines are equal to each other.

18. In the base of a △ find a point such that the st. lines drawn from that point || to the sides of the △ and terminated by the sides are equal to each other.

19. Give a proof for the following method of drawing a line through **P** || **AB**;—



Place the set-square with the hypotenuse along the st. line **AB.**

Place a ruler against another side of the set-square as in the diagram.

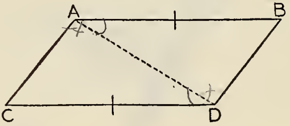
Hold the ruler firmly in position and slide the set-square along it until the hypotenuse comes to the point **P.**

A line drawn through **P** along the hypotenuse of the set-square is || **AB.**

PARALLELOGRAMS

PROPOSITION 6 THEOREM

Straight lines which join the ends of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Hypothesis.—AB, CD are = and ||.

Prove that (1) AC = BD,

(2) AC || BD.

Construction.—Join AD.

Proof.— ∵ AB || CD,

and AD is a transversal,

∴ ∠ BAD = ∠ CDA. (II—2.)

In △s BAD, CDA, $\left\{\begin{array}{c}\text{BA} = \text{CD},\\ \text{AD is common},\\ \angle\ \text{BAD} = \angle\ \text{CDA},\end{array}\right.$

$\left.\begin{array}{r}\therefore\ \text{BD} = \text{AC},\\ \text{and}\ \angle\ \text{BDA} = \angle\ \text{CAD},\end{array}\right\}$ (I—1.)

∵ transversal AD

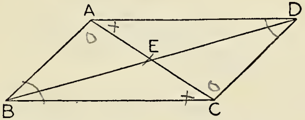
makes ∠ BDA = ∠ CAD,

∴ BD || AC. (II—3.)

Proposition 7 Theorem

In any parallelogram:

(1) The opposite sides are equal;
(2) The opposite angles are equal;
(3) The diagonal bisects the area;
(4) The diagonals bisect each other.



Hypothesis.—ABCD is a ||gm, AC, BD its diagonals.

Prove that (1) AD = BC and AB = CD.

(2) ∠ BAD = ∠ BCD and ∠ ABC = ∠ ADC.

(3) △ ABC = △ ACD.

(4) AE = EC and BE = ED.

Proof.—∵ AC cuts || lines AD, BC,

∴ ∠ DAC = ∠ ACB. (II—2.)

∵ AC cuts || lines DC, AB,

∴ ∠ DCA = ∠ CAB.

In △s ACD, ACB, { ∠ DAC = ∠ ACB, ∠ DCA = ∠ CAB, AC is common,

∴ (1) AD = BC, and CD = AB,
(2) also ∠ ADC = ∠ ABC,
(3) and △ ADC = △ ABC. } (I—4.)

Similarly it may be shown that ∠ BAD = ∠ BCD.

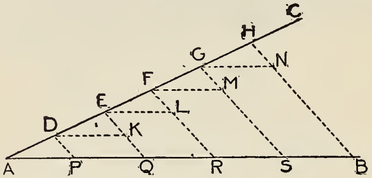
In △s AED, BEC, { AD = BC, ∠ DAE = ∠ BCE, ∠ ADE = ∠ CBE,

(4) ∴ AE = EC,
and DE = EB. } (I—4.)

Construction

Proposition 8 Problem

Divide a straight line into any number of equal parts.



Let AB be the given st. line.

To divide AB into five equal parts.

Construction.—From A draw a st. line AC.

From AC cut off five equal parts AD, DE, EF, FG, GH.

Join HB.

Through D, E, F, G draw lines || HB cutting AB at P, Q, R, S.

AB is divided into five equal parts at P, Q, R, S.

Proof.—Through D, E, F, G draw DK, EL, FM, GN || AB.

$\because$ AE cuts the parallels AP, DK,

$\therefore$ $\angle$ EDK = $\angle$ DAP. (Int. 41.)

$\because$ AE cuts the parallels DP, EQ,

$\therefore$ $\angle$ ADP = $\angle$ DEQ.

In $\triangle$s ADP, DEK, $\left\{\begin{array}{l}\angle \text{DAP} = \angle \text{EDK}, \\ \angle \text{ADP} = \angle \text{DEK}, \\ \text{AD} = \text{DE},\end{array}\right.$

$\therefore$ AP = DK (I—4.)

But PQ = DK. (II—7.)

$\therefore$ PQ = AP.

Similarly it may be shown that each of **QR**, **RS**, **SB** = **AP**.

By this method a st. line may be divided into any number of equal parts.

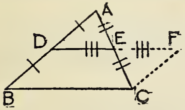
85.—Exercises

1. The diagonals of a rectangle are equal to each other.

2. Make a ||gm having one diagonal 7 cm., the other diagonal 6 cm., and an ∠ between the diagonals 60°. Measure the sides of the ||gm. (Ans. 33 mm. and 56 mm.)

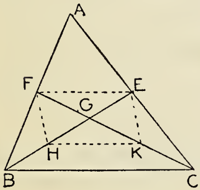
3. Make a ||gm having one diagonal 8 cm., the other 10 cm. and one side 6 cm. Measure the ∠s between the diagonals. (Ans. 83° and 97°.)

4. The st. line joining the middle points of the sides of a △ is || the base, and equal to half of it.



Note.—**D**, **E** *are the middle points of* **AB**, **AC**. *Produce* **DE** *to* **F** *making* **EF** = **DE**. *Join* **FC**.

5. Of two medians of a △ each cuts the other at the point of trisection remote from the vertex.



Note.—Medians **BE**, **CF** *cut at* **G**. *Bisect* **BG**, **CG** *at* **H**, **K**. *Join* **FH**, **HK**, **KE**, **EF**.

6. The medians of a △ pass through one point.

Definition.—The point where the medians of a △ intersect is called the **centroid** of the △.

7. A st. line drawn through the middle point of one side of a △, || to a second side, bisects the third side.

8. In any ||gm the diagonal which joins the vertices of the obtuse ∠s is shorter than the other diagonal.

9. If two sides of a quadrilateral are ||, and the other two are equal to each other but not ||, the diagonals of the quadrilateral are equal.

10. Through a given point draw a st. line, such that the part of it intercepted between two given || st. lines is equal to a given st. line.

Show that, in general, two such lines can be drawn.

11. Through a given point draw a st. line that shall be equidistant from two other given points.

Show that, in general, two such lines can be drawn.

12. Draw a st. line || to a given st. line, and such that the part of it intercepted between two given intersecting lines is equal to a given st. line.

13. **BAC** is a given $\angle$, and **P** is a given point. Draw a st. line terminated in the st. lines **AB, AC** and bisected at **P.**

14. Construct a $\triangle$ having given the middle points of the three sides.

15. Every st. line drawn through the intersection of the diagonals of a ||gm, and terminated by a pair of opposite sides, is bisected, and bisects the ||gm.

16. Bisect a given ||gm by a st. line drawn through a given point.

17. Divide a given st. line in two parts so that one part is twice as long as the other.

18. The bisectors of two opposite $\angle$s of a ||gm are || to each other.

19. In the quadrilateral **ABCD, AB** || **CD** and **AD, BC** are = but not ||. Prove that (1) $\angle$ **C** = $\angle$ **D**; (2) if **E, F** are the middle points of **AB, CD** respectively, **EF** $\perp$ **AB.**

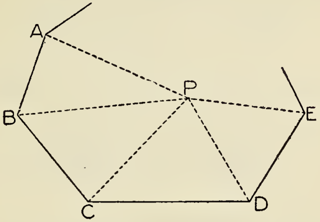
20. On a given st. line construct a square.

21. Construct a square having its diagonal equal to a given st. line.

22. **ABC** is a $\triangle$ and **DE** a st. line. Draw a st. line = **DE,** || **BC** and terminated in **AB, AC,** or in these lines produced.

PROPOSITION 9 THEOREM

The sum of the interior angles of a polygon of n sides is $(2n-4)$ right angles.



Hypothesis.—ABCDE, etc., is a closed polygon of n sides.

Prove that the sum of the interior angles is $(2n - 4)$ rt. $\angle$s.

Construction.—Take any point P within the polygon and join P to the vertices.

Proof.—The polygon is divided into n $\triangle$s PAB, PBC, PCD, etc.

The sum of the interior $\angle$s of each $\triangle$ is two rt. $\angle$s. (Int. 49.)

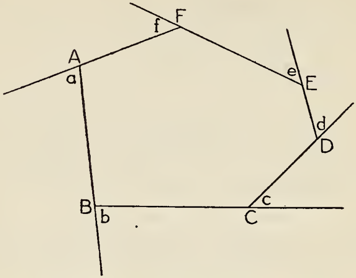
$\therefore$ the sum of the $\angle$s of the n $\triangle$s is $2n$ rt. $\angle$s.

But the $\angle$s of the n $\triangle$s make up the interior $\angle$s of the polygon together with the $\angle$s about the point P.

And the sum of the $\angle$s about P equals 4 rt. $\angle$s.

$\therefore$ the sum of the interior $\angle$s of the polygon $=$ $(2n - 4)$ rt. $\angle$s.

Cor.—If the sides of a polygon are produced in order, the sum of the exterior angles thus formed is four right angles.



If the polygon has n sides, the sum of all the st. $\angle$s at the vertices $= 2n$ rt. $\angle$s.

But, the sum of the interior $\angle$s $= (2n-4)$ rt. $\angle$s. (II—9.)

$\therefore$, subtracting, $\angle a + \angle b +$ etc. $= 4$ rt. $\angle$s.

86.—Exercises

1. Find the number of degrees in an exterior $\angle$ of an equiangular polygon of twelve sides.

Hence, find the number of degrees in each interior $\angle$.

2. Find the number of degrees in each $\angle$ of (a) an equiangular pentagon; (b) an equiangular hexagon; (c) an equiangular octagon; (d) an equiangular decagon.

3. Each $\angle$ of an equiangular polygon contains 162°. Find the number of sides.

4. Each $\angle$ of an equiangular polygon contains 170°. Find the number of sides.

5. Show that the space around a point may be exactly filled in by six equilateral $\triangle$s, four squares, or three equiangular hexagons. Draw the diagram in each case.

Areas of Parallelograms and Triangles

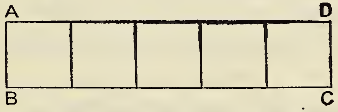
87. A square unit of area is a square, each side of which is equal to a unit of length.

Examples:—A square inch is a square each side of which is one inch; a square centimetre is a square each side of which is one centimetre.

The acre is an exceptional case.

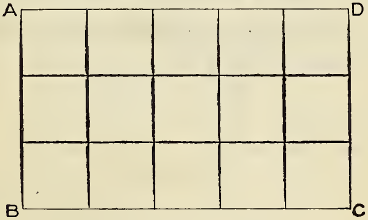
A numerical measure of any area is the number of times the area contains some unit of area.

ABCD is a rectangle one centimetre wide and five centimetres long.



This rectangle is a strip divided into five square centimetres, and consequently the numerical measure of its area in square centimetres is 5.

ABCD is a rectangle 3 cm. wide and 5 cm. long.



This rectangle is divided into 5 strips of 3 sq. cm. each, or into 3 strips of 5 sq. cm. each, and consequently the measure of the area in square centimetres is 5×3 sq. cm., 3×5 sq. cm.

Similarly, if the length of a rectangle is 2·34 inches and its breadth ·56 of an inch, the one-hundredth of an inch may be taken as the unit and the rectangle can be divided into 234 strips, each containing 56 square one-hundredths of an inch. The measure of the area, then, is 234×56 of these small squares, ten thousand (100×100) of which make one square inch.

This method of expressing the area of a rectangle may be carried to any degree of approximation, so that in all cases the numerical measure of its area is equal to the product of its length by its breadth.

In a rectangle any side may be called the base, and so either of the adjacent sides is the altitude.

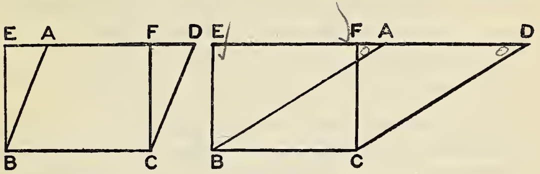
A rectangle, as ABCD, is commonly represented by the symbol AB, BC, where AB and BC may be taken to represent the number of units in the length and the breadth respectively.

Or, if a be the measure of the base of a rectangle and b the measure of its altitude, the area is ab.

In the case of a square, the base is equal to the altitude, and if the measure of each be a, the area is a^2.

Proposition 10 Theorem

The area of a parallelogram is equal to that of a rectangle on the same base and of the same altitude.



Hypothesis.—ABCD is a ||gm and EBCF a rectangle on the same base BC and of the same altitude EB.

Prove that the area of the ||gm ABCD = the area of rect. EBCF.

Proof.— $\because$ ED cuts the ||s AB, DC,

$\therefore \angle$ EAB = $\angle$ FDC. (Int. 41.)

$\because$ ABCD is a ||gm,

$\therefore$ AB = CD. (II—7.)

In $\triangle$s EAB, FDC, $\begin{cases} \angle \text{EAB} = \angle \text{FDC}, \\ \angle \text{AEB} = \angle \text{DFC}, \\ \text{AB} = \text{DC}, \end{cases}$

$\therefore \triangle$ AEB = $\triangle$ FDC, (I—4.)

Figure EBCD − $\triangle$ EAB = ||gm ABCD,

Figure EBCD − $\triangle$ FDC = rect. EBCF;

and as equal parts have been taken from the same area, the remainders are equal.

$\therefore$ ||gm ABCD = rect. EBCF.

Cor.—If *a* be the measure of the base of a ||gm and *b* the measure of its altitude, the area, being the same as that of a rect. of the same base and altitude, = *ab*.

88.—Practical Exercises

1. Draw a ||gm having two adjacent sides 6·4 cm. and 7·3 cm. and the contained ∠ 30°. Find its area.

2. Draw a ||gm having the two diagonals 4·8 cm. and 6·8 cm. and an ∠ between the diagonals 75°. Find its area.

3. The area of a ||gm is 50 sq. cm., one side is 10 cm. and one ∠ is 60°. Construct the ||gm, and measure the other side.

4. Draw a rectangle of base 7 cm. and height 4 cm. On the same base construct a ||gm having the same area as the rectangle and two of its sides each 65 mm. Measure one of the smaller ∠s of the ||gm.

5. Make a ||gm having sides 10 and 7 cm. and one ∠ 60°. Make a rhombus equal in area to the ||gm and having each side 10 cm. Measure the shorter diagonal of the rhombus.

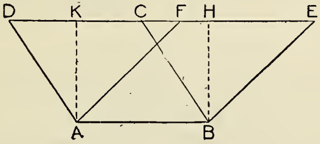
6. Make a rectangle 8 cm. by 5 cm. Construct a ||gm equal in area to the rectangle and having two sides 7 cm. and 8 cm. Construct a rhombus equal in area to the ||gm and having each side 7 cm. Measure the shorter diagonal of the rhombus.

7. Make a rhombus having each side 8 cm. and its area 50 sq. cm. Measure the shorter diagonal.

Answers:—1. 23·4 sq. cm. nearly. 2. 15·8 sq. cm. nearly. 3. 57·7 mm. nearly. 4. 38° nearly. 5. 64 mm. nearly. 6. 64 mm. nearly. 7. 69 mm. nearly.

PROPOSITION 11 THEOREM

Parallelograms on the same base and between the same parallels are equal in area.



Hypothesis.—ABCD, ABEF are ||gms on the same base AB and between the same ||s AB, DE.

Prove that ||gm ABCD = ||gm ABEF.

Construction.—Draw AK, BH each ⊥ to both AB and DE.

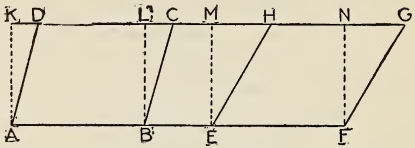
Proof.—∵ ||gm ABCD = rect. ABHK, (II—10.)

and ||gm ABEF = rect. ABHK,

∴ ||gm ABCD = ||gm ABEF.

PROPOSITION 12 THEOREM

Parallelograms on equal bases and between the same parallels are equal in area.



Hypothesis.—ABCD, EFGH are ||gms on the equal bases AB, EF and between the same ||s AF, DG.

Prove that ||gm ABCD = ||gm EFGH.

Construction.—Draw AK, BL, EM, FN each ⊥ to both AF, DG.

Proof.— ∵ AB = EF,

and AK = EM, (II—7.)

∴ rect. KB = rect. MF.

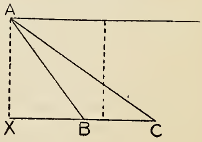
But ||gm ABCD = rect. KB, (II—10.)

and ||gm EFGH = rect. MF,

∴ ||gm ABCD = ||gm EFGH.

89. Draw an acute-∠d △ ABC. Draw the ⊥ from A to BC. Draw through A, a st. line ||BC. Show that the ⊥ distance between these || lines at any place = the altitude of △ ABC.

Draw an obtuse-∠d △ ABC, having the obtuse ∠ at B. Draw the altitude AX. Show that it falls without the △. Draw through A, a st. line || BC. Show that the distance between these || lines at any place = the altitude of the △.

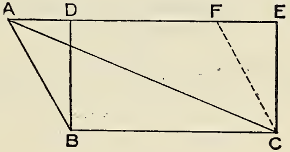


Taking **C** as the vertex and **AB** as the base, draw the altitude.

If a △ be between two ||s, having its base in one of the ||s and its vertex in the other, its altitude is the distance between the ||s.

PROPOSITION 13 THEOREM

The area of a triangle is half that of the rectangle on the same base and of the same altitude as the triangle.



Hypothesis.—**ABC** is a △ and **DBCE** a rectangle on the same base and of the same altitude **BD**.

Prove that area of △ **ABC** = half that of rect. **DBCE**.

Construction.—Through **C** draw **CF** || **BA**.

Proof.—∵ **AC** is a diagonal of ||gm **ABCF**,

∴ △ **ABC** = half of ||gm **ABCF**. (II—7.)

But ||gm **ABCF** = rect. **DBCE**, (II—10.)

∴ △ **ABC** = half of rect. **DBCE**.

Cor.—If a be the measure of the base of a △ and b the measure of its altitude, the measure of its area is $\frac{1}{2}ab$.

90.—Practical Exercises

1. Draw a rt.-∠d △ having the sides that contain the right ∠ 56 mm. and 72 mm. Find the area of the △.

2. Make a △ **ABC**, having $b = 6$ cm., $c = 8$ cm., and ∠ **A** = 72° Find its area.

3. Draw a △ having its sides 73 mm., 57 mm. and 48 mm. Find its area.

4. Find the area of the △: $a = 10$ cm., ∠ **B** = 42°, ∠ **C** = 58°.

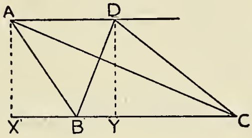
5. The sides of a triangular field are 36 chains, 25 chains and 29 chains. Draw a diagram and find the number of acres in the field. (Scale: 1 mm. to the chain.)

6. Two sides of a triangular field are 41 and 38 chains and the contained ∠ is 70°. Find its area in acres.

Answers:—1. 20·16 sq. cm.; 2. 23 sq. cm. nearly; 3. 13·7 sq. cm. nearly; 4. 28·8 sq. cm.; 5. 36 ac.; 6. 73 ac. nearly.

PROPOSITION 14 THEOREM

If two triangles are on the same base and between the same parallels, the triangles are equal in area.



Hypothesis.—ABC, DBC are $\triangle$s on the same base BC and between the same $\|$s AD, BC.

Prove that $\triangle$ ABC = $\triangle$ DBC.

Construction.—Draw AX, DY $\perp$ BC.

Proof.— $\triangle$ ABC = $\frac{1}{2}$ rect. AX.BC. (II—13.)

$\triangle$ DBC = $\frac{1}{2}$ rect. DY.BC.

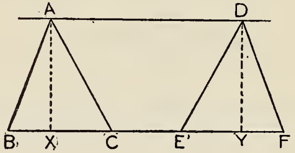
But, $\because$ AX = DY,

$\therefore$ rect. AX.BC = rect. DY.BC.

and $\therefore$ $\triangle$ ABC = $\triangle$ DBC.

Proposition 15 Theorem

If two triangles are on equal bases and between the same parallels, the triangles are equal in area.



Hypothesis.—ABC, DEF are $\triangle$s on equal bases BC, EF and between the same $\|$s AD, BF.

Prove that $\triangle$ ABC = $\triangle$ DEF.

Construction.—Draw AX, DY $\perp$ BF.

Proof.— $\triangle$ ABC = $\frac{1}{2}$ rect. AX.BC. (II—13.)

$\triangle$ DEF = $\frac{1}{2}$ rect. DY.EF.

But, $\because$ BC = EF,

and AX = DY, (II—7.)

$\therefore$ rect. AX.BC = rect. DY.EF.

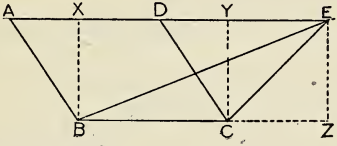
Hence, $\triangle$ ABC = $\triangle$ DEF.

Cor. 1.—**Triangles on equal bases and of the same altitude are equal in area.**

Cor. 2.—**A median bisects the area of the triangle.**

Proposition 16 Theorem

If a parallelogram and a triangle are on the same base and between the same parallels, the parallelogram is double the triangle.



Hypothesis.—**ABCD** is a ||gm and **EBC** a △ on the same base **BC** and between the same ||s **AE, BC.**

Prove that ||gm **ABCD** = twice △ **EBC.**

Construction.—Draw **BX, CY, EZ** ⊥ **BC** and **AE.**

Proof.—||gm **ABCD** = rect. **BX.BC.** (II—10.)

△ **EBC** = $\frac{1}{2}$ rect. **EZ.BC.** (II—13)

But, ∵ **BX** = **EZ,** (II—7.)

∴ rect. **BX.BC** = rect. **EZ.BC.**

And ∴ ||gm **ABCD** = twice △ **EBC.**

91.—Exercises

1. △s **ABC, DEF** are between the same ||s **AD** and **BCEF,** and **BC** > **EF.** Prove that △ **ABC** > △ **DEF.**

2. On the same base with a ||gm construct a rectangle equal in area to the ||gm.

3. On the same base with a given ||gm, construct a ||gm equal in area to the given ||gm, and having one of its sides equal to a given st. line.

4. Construct a rect. equal in area to a given ||gm, and having one of its sides equal to a given st. line.

5. Make a ||gm with sides 5 cm. and 3 cm., and contained ∠ 125°. Construct an equivalent rect. having one side 1·5 cm.

6. On the same base as a given △ construct a rect. equal in area to the △.

7. Construct a rect. equal in area to a given △, and having one of its sides equal to a given st. line.

8. On the same base with a ||gm construct a rhombus equal in area to the ||gm.

9. Construct a rhombus equal in area to a given ||gm, and having each of its sides equal to a given st. line.

10. On the same base with a given △, construct a rt.-∠d △ equal in area to the given △.

11. On the same base with a given △, construct an isosceles △ equal in area to the given △.

12. If, in the ||gm **ABCD**, **P** be any point between **AB**, **CD** produced indefinitely, the sum of the △s **PAB**, **PCD** equals half the ||gm; and if **P** be any point not between **AB**, **CD**, the difference of the △s **PAB**, **PCD** equals half the ||gm.

13. **AB** and **ECD** are two || st. lines; **BF**, **DF** are drawn || **AD**, **AE** respectively; prove that △s **ABC**, **DEF** are equal to each other.

14. On the same base with a given △, construct a △ equal in area to the given △, and having its vertex in a given st. line.

15. If two △s have two sides of one respectively equal to two sides of the other and the contained ∠s supplementary, the △s are equal in area.

16. **ABCD** is a ||gm, and **P** is a point in the diagonal **AC**. Prove that △ **PAB** = △ **PAD**.

17. **P** is a point within a ||gm **ABCD**. Prove that △ **PAC** equals the difference between △s **PAB**, **PAD**.

18. In △ **ABC**, **BC** and **CA** are produced to **P** and **Q** respectively, such that **CP** = one-half of **BC**, and **AQ** = one-half of **CA**. Show that △ **QCP** = three-fourths of △ **ABC**.

19. The medians **BE**, **CD** of the △ **ABC** intersect at **F**. Show that △ **BFC** = quadrilateral **ADFE**.

20. On the sides **AB**, **BC** of a △ the ||gms **ABDE**, **CBFG** are described external to the △. **ED** and **GF** meet at **H** and **BH** is joined. On **AC** the ||gm **CAKL** is described with **CL** and **AK** || and = **HB**. Prove ||gm **AL** = ||gm **AD** + ||gm **CF**.

21. Two △s are equal in area and between the same ||s. Prove that they are on equal bases.

22. Of all △s on a given base and between the same ||s, the isosceles △ has the least perimeter.

23. **ABCD** is a ||gm, and **E** is a point such that **AE**, **CE** are respectively ⊥ and || to **BD**. Show that **BE** = **CD**.

24. The side **AB** of ||gm **ABCD** is produced to **E** and **DE** cuts **BC** at **F**. **AF** and **CE** are joined. Prove that △ **AFE** = △ **CBE**.

25. In the quadrilateral **ABCD**, **AB** || **CD**. If **AB** = a, **CD** = b and the distance between **AB** and **CD** = h, show that the area of **ABCD** = $\frac{1}{2} h (a + b)$.

26. Two sides **AB**, **AC** of a △ are given in length, find the ∠ **A** for which the area of the △ will be greatest.

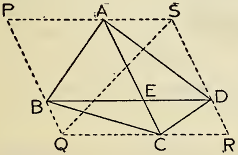
27. The medians **AD**, **BE** of △ **ABC** intersect at **G**, and **CG** is joined. Prove that the three lines **AG**, **BG**, **CG** trisect the area of the △.

28. Bisect the area of a △ by a st. line drawn through a vertex.

29. Trisect the area of a △ by two st. lines drawn through a vertex.

30. Bisect the area of a △ by a st. line drawn through a given point in one of the sides.

31. Trisect the area of a △ by two st. lines drawn through a given point in one of the sides.



32. The area of any quadrilateral **ABCD** is equal to that of a △ having two sides and their included ∠ respectively equal to the diagonals of the quadrilateral and their included ∠.

NOTE.—*Draw* **PS** *and* **QR** || **BD**, **PQ** *and* **SR** || **AC**. *Join* **SQ**.

33. Prove that in a rhombus the distance between one pair of opposite sides equals the distance between the other pair.

34. ||gms are described on the same base and between the same ||s. Find the locus of the intersection of their diagonals.

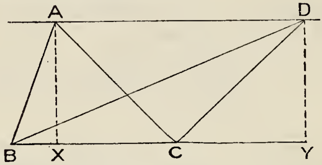
35. Prove that the area of a rhombus is half the product of the lengths of its diagonals.

36. ABCD is a quadrilateral in which AB || CD, E is the middle point of AD. Prove that △ BEC = ½ quadrilateral ABCD.

37. Divide a given △ into seven equal parts.

PROPOSITION 17 THEOREM

If two equal triangles are on the same side of a common base, the straight line joining their vertices is parallel to the common base.



Hypothesis.—ABC, DBC are two equal △s on the same side of the common base BC.

Prove that AD || BC.

Construction.—Draw AX and DY ⊥ BC.

Proof.— △ ABC = ½ rect. BC.AX. (II—13.)

△ DBC = ½ rect. BC.DY;

but △ ABC = △ DBC,

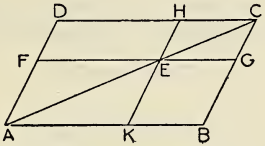
∴ ½ rect. BC.AX = ½ rect. BC.DY

and hence AX = DY,

that is, AX and DY are both = and || to each other

∴ AD || XY. (II—6.)

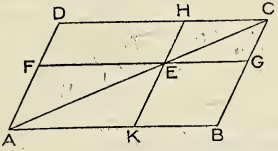
92. If, through any point **E**, in the diagonal **AC** of a parallelogram **BD**, two straight lines **FEG**, **HEK** are drawn parallel respectively to the sides **DC**, **DA** of the parallelogram,



the ||gms **FK** and **HG** are said to be parallelograms about the diagonal **AC**, and the ||gms **DE**, **EB** are called the **complements** of the ||gms **FK**, **HG**, which are about the diagonal.

PROPOSITION 18 THEOREM

The complements of the parallelograms about the diagonal of any parallelogram are equal to each other.



Hypothesis.—**FK** and **HG** are ||gms about the diagonal **AC** of the ||gm **ABCD**.

Prove that the complements **DE**, **EB** are equal to each other.

Proof.— $\because$ **AE** is a diagonal of ||gm **FK**,

$\therefore \triangle$ **AFE** = $\triangle$ **AKE**. (II—7.)

Similarly $\triangle$ HEC = $\triangle$ EGC.

$\therefore$ $\triangle$ AFE + $\triangle$ HEC = $\triangle$ AKE + $\triangle$ EGC.

But, $\because$ AC is a diagonal of ||gm ABCD

$\therefore$ $\triangle$ ADC = $\triangle$ ABC.

$\therefore$ $\triangle$ ADC − ($\triangle$ AFE + $\triangle$ HEC)

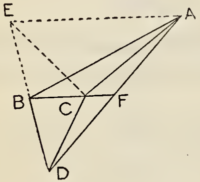
= $\triangle$ ABC − ($\triangle$ AKE + $\triangle$ EGC).

$\therefore$ ||gm DE = ||gm EB.

93.—Exercises

1. If two equal $\triangle$s are on equal segments of the same st. line and on the same side of the line, the st. line joining their vertices is || to the line containing their bases.

2. Through P, a point within the ||gm ABCD, EPF is drawn || AB and GPH is drawn || AD. If ||gm AP = ||gm PC, show that P is on the diagonal BD. (Converse of II—18.)



3. Two equal $\triangle$s ABC, DBC are on opposite sides of the same base. Prove that AD is bisected by BC, or BC produced.

Note.—*Produce* DB *making* BE = DB. *Join* EA, EC.

Give another proof of this proposition using $\perp$s from A and D to BC and II—13.

4. The median drawn to the base of a $\triangle$ bisects all st. lines drawn || to the base and terminated by the sides, or the sides produced.

5. P is a point within a $\triangle$ ABC and is such that $\triangle$ PAB + $\triangle$ PBC is constant. Prove that the locus of P is a st. line || AC.

6. ||gms about the diagonal of a square are squares.

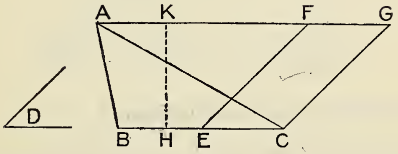
7. D, E, F are respectively the middle points of the sides BC, CA, AB in the $\triangle$ ABC. Prove $\triangle$ BEF = $\triangle$ CEF and hence that EF || BC.

8. In the diagram of II—18, show that FK || HG.

CONSTRUCTIONS

PROPOSITION 19 PROBLEM

Construct a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.



Let ABC be the given $\triangle$ and D the given $\angle$.

It is required to construct a ||gm equal in area to $\triangle$ ABC and having one $\angle$ equal to $\angle$ D.

Construction.—Through A draw AFG || BC. Bisect BC at E. At E make $\angle$ CEF = $\angle$ D. Through C draw CG || EF.

FC is the required ||gm.

Proof.—Draw any line HK $\perp$ to the two || st. lines. HK is the common altitude of the ||gm FC and the $\triangle$ ABC.

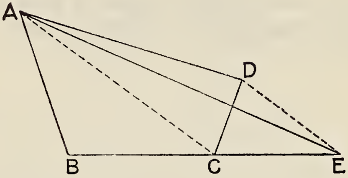
$$\|\text{gm } FC = \text{rect. } EC.HK. \qquad \text{(II—10.)}$$

$$= \tfrac{1}{2} \text{ rect. } BC.HK, \because EC = \tfrac{1}{2} BC,$$

$$= \triangle ABC. \qquad \text{(II—13.)}$$

Proposition 20 Problem

Construct a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

It is required to construct a $\triangle$ equal in area to ABCD.

Construction.—Join AC. Through D draw DE $\parallel$ AC and meeting BC produced at E. Join AE.

$$\triangle \text{ ABE} = \text{quadrilateral ABCD.}$$

Proof.—$\because$ DE $\parallel$ AC,

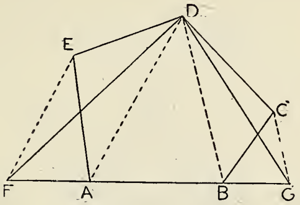
$\therefore$ $\triangle$ EAC = $\triangle$ DAC. (II—14.)

To each of these equals add $\triangle$ ABC.

Then $\triangle$ ABE = quadrilateral ABCD.

PROPOSITION 21 PROBLEM

Construct a triangle equal in area to a given rectilineal figure.



Let the pentagon ABCDE be the given rectilineal figure.

Construction.—Join AD, BD. Through E, draw EF || AD and meeting BA at F. Through C draw CG || BD and meeting AB at G.

Join DF, DG.

$\triangle$ DFG = figure ABCDE.

Proof.— $\because$ EF || AD,

$\therefore$ $\triangle$ DFA = $\triangle$ DEA. (II—14.)

$\because$ CG || DB,

$\therefore$ $\triangle$ DGB = $\triangle$ DCB.

$\therefore$ $\triangle$ DFA + $\triangle$ DAB + $\triangle$ DBG

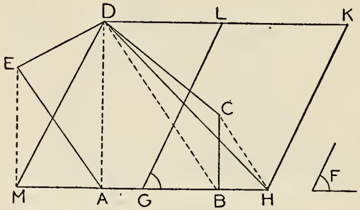
= $\triangle$ DEA + $\triangle$ DAB + $\triangle$ DCB;

i.e., $\triangle$ DFG = figure ABCDE.

By this method a $\triangle$ may be constructed equal in area to a given rectilineal figure of any number of sides; *e.g.*, for a figure of seven sides, an equivalent figure of five sides may be constructed, and then, as in the construction just given, a $\triangle$ may be constructed equal to the figure of five sides.

Proposition 22 Problem

Describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given angle.



Let **ABCDE** be the given rectilineal figure and **F** the given $\angle$.

It is required to construct a $\|$gm = **ABCDE**, and having an $\angle$ = $\angle$ **F**.

Construction.—Make $\triangle$ **DMH** equal in area to figure **ABCDE**. (II—21.)

Make $\|$gm **LGHK** = $\triangle$ **DMH**, and having $\angle$ **LGH** = $\angle$ **F**. (II—19.)

Then $\|$gm **LGHK** = figure **ABCDE**, and has $\angle$ **LGH** = $\angle$ **F**.

94.—Exercises

1. Construct a rect. equal in area to a given $\triangle$.

2. Construct a rect. equal in area to a given quadrilateral.

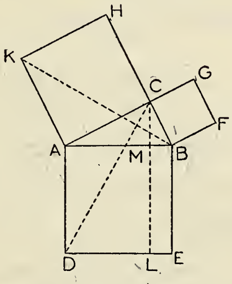
3. Construct a quadrilateral equal in area to a given hexagon.

4. On one side of a given $\triangle$ construct a rhombus equal in area to the given $\triangle$.

5. Construct a $\triangle$ equal in area to a given $\|$gm, and having one of its $\angle$s = a given $\angle$.

PROPOSITION 23 THEOREM

The square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.



Hypothesis.—ABC is a $\triangle$ in which $\angle$ ACB is a rt. $\angle$, and AE, BG, CK are squares on AB, BC and CA.

Prove that $AB^2 = AC^2 + BC^2$.

Construction.—Through C draw CL || AD.

Join KB, CD.

Proof.—$\because$ $\angle$s HCA, ACB, BCG are rt. $\angle$s,

$\therefore$ $\angle$s HCB, ACG are st. $\angle$s.

and $\therefore$ HCB, ACG are st. lines.

$\angle$ BAD $=$ $\angle$ KAC,

to each add $\angle$ CAB,

then $\angle$ CAD $=$ $\angle$ KAB.

$$\text{In } \triangle\text{s CAD, KAB, } \begin{cases} \text{CA} = \text{KA} \\ \text{AD} = \text{AB} \\ \angle\,\text{CAD} = \angle\,\text{KAB} \end{cases}$$

$\therefore$ $\triangle$ CAD $=$ $\triangle$ KAB (I—1.)

∵ rect. ADLM and △ CAD are on the same base AD and between the same ||s CL, AD,

∴ rect. AL = twice △ CAD. (II—16.)

Similarly, sq. HA = twice △ KAB.

∴ rect. AL = sq. HA.

In the same manner, by joining CE and AF, it may be shown that

rect. BL = sq. BG.

∴ rect. AL + rect. BL = sq. HA + sq. BG,

i.e., $AB^2 = AC^2 + BC^2$.

95. Many proofs have been given for this important theorem. Pythagoras (582 to 500 B.C.) is said by tradition to have been the first to prove it, and from that it is commonly called the Theorem of Pythagoras, or the Pythagorean Theorem. The proof given above is attributed to Euclid (about 300 B.C.). An alternative proof is given in Book IV.

96.—Exercises

1. Draw two st. lines 5 cm. and 6 cm. in length. Describe squares on both, and make a square equal in area to the two squares. Measure the side of this last square and check your result by calculation.

2. Draw three squares having sides 1 in., 2 in. and $2\frac{1}{2}$ in. Make one square equal to the sum of the three. Check by calculation.

3. Draw two squares having sides $1\frac{1}{2}$ in. and $2\frac{1}{2}$ in. Make a third square equal to the difference of the first two. Check by calculation.

4. Draw two squares having sides 9 cm. and 6 cm. Make a third square equal to the difference of the first two. Check your result by calculation.

5. Draw any square and one of its diagonals. Draw a square on the diagonal and show that it is double the first square.

6. Draw a square having each side 4 cm. Draw a second square double the first. Measure a side, and check by calculation.

7. Draw a square having one side 45 mm. Draw a second square three times the first. Measure its side, and check by calculation.

8. Draw three lines in the ratio 1:2:3. Draw squares on the lines, and divide the two larger so as to show that the squares are in the ratio 1:4:9.

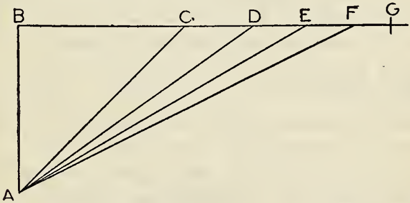
9. Draw a st. line $\sqrt{2}$ in. in length.

10. Draw a st. line $\sqrt{3}$ in. in length.

11. Draw a st. line $\sqrt{5}$ in. in length.

12. Draw any rt.-∠d △. Describe equilateral △s on the three sides. Find the areas of the △s and compare that on the hypotenuse with the sum of those on the other two sides.

13.



AB is one inch in length, ∠ **B** a rt. ∠, **BC** is one inch **BD** is cut off = **AC**, **BE** = **AD**, **BF** = **AE**, **BG** = **AF**, etc. Show that **BD** = $\sqrt{2}$ in., **BE** = $\sqrt{3}$ in., **BF** = $\sqrt{4} = 2$ in., **BG** = $\sqrt{5}$ in., etc.

14. Construct a square equal to half a given square.

15. If a ⊥ be drawn from the vertex of a △ to the base, the difference of the squares on the segments of the base = the difference of the squares on the other two sides.

Hence, prove that the altitudes of a △ pass through one point.

16. **A** is a given st. line. Find another st. line **B**, such that the difference of the square on **A** and **B** may be equal to the difference of two given squares.

17. If the diagonals of a quadrilateral cut at rt. ∠s, the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair.

18. The sum of the squares on the diagonals of a rhombus equals the sum of the squares on the four sides.

19. Five times the square on the hypotenuse of a rt.-∠d △ equals four times the sum of the squares on the medians drawn to the other two sides.

20. In an isosceles rt.-∠d △ the sides have the ratios $1:1:\sqrt{2}$.

21. If the angles of a △ are 90°, 30°, 60°, the sides have the ratios $2:1:\sqrt{3}$.

22. Divide a st. line into two parts such that the sum of the squares on the parts equals the square on another given st. line. When is this impossible?

23. In the st. line **AB** produced find a point **C** such that the sum of the squares on **AC**, **BC** equals the square on a given st. line.

24. Divide a given st. line into two parts such that the square on one part is double the square on the other part.

25. **ABCD** is a rect., and **P** is any point. Show that $PA^2 + PC^2 = PB^2 + PD^2$.

26. **ABC** is a △ rt.-∠d at **A**. **E** is a point on **AC** and **F** is a point on **AB**. Show that $BE^2 + CF^2 = EF^2 + BC^2$.

27. If two rt.-∠d △s have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the △s are congruent.

28. The square on the side opposite an acute ∠ of a △ is less than the sum of the squares on the other two sides.

29. The square on the side opposite an obtuse ∠ of a △ is greater than the sum of the squares on the other two sides.

30. Construct a square that contains 20 square inches.

31. In the diagram of II—23, show that **KB**, **CD** cut at rt. ∠s.

32. In the diagram of II—23, if **KD** be joined, show that △ **KAD** = △ **ABC**.

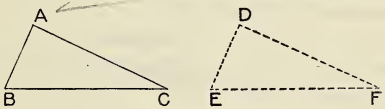
33. In the diagram of II—23, the distance of **E** from **AC** = **AC** + **CB**.

34. **ABC** is an isosceles rt.-∠d △ in which **C** is the rt. ∠. **CB** is produced to **D** making **BD** = **CB**. ⊥s to **AB**, **BD** at **A**, **D** respectively meet at **E**. Prove that **AE** = 2 **AB**.

PROPOSITION 24 THEOREM

(Converse of II—23)

If the square on one side of a triangle be equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.



Hypothesis.—ABC is a $\triangle$ in which $BC^2 = AB^2 + AC^2$.

Prove that $\angle$ A is a rt. $\angle$.

Construction.—Make a rt. $\angle$ D and cut off DE = AB, DF = AC.

Join EF.

$$BC^2 = AB^2 + AC^2 \qquad (Hyp.)$$

$$= DE^2 + DF^2$$

$$= EF^2 \ (\because \text{D is a rt. } \angle). \qquad (\text{II—23.})$$

$\therefore$ BC = EF.

In $\triangle$s ABC, DEF, $\left\{\begin{array}{l} AB = DE, \\ AC = DF, \\ BC = EF, \end{array}\right.$

$\therefore \angle A = \angle D$. (I—3.)

$\therefore \angle$ A is a rt. $\angle$.

97.—Exercises

1. The sides of a △ are 3 in., 4 in. and 5 in. Prove that it is a rt.-∠d △.

2. The sides of a △ are 13 mm., 84 mm. and 85 mm. Prove that it is a rt.-∠d △.

3. In the quadrilateral **ABCD**, $\mathbf{AB}^2 + \mathbf{CD}^2 = \mathbf{BC}^2 + \mathbf{AD}^2$. Prove that the diagonals **AC**, **BD** cut at rt. ∠s.

4. If the sq. on one side of a △ be less than the sum of the squares on the other two sides, the ∠ contained by these two sides is an acute ∠. (Converse of § 96, Ex. 28.)

5. State and prove a converse of § 96, Ex. 29.

6. Using a tape-measure, or a knotted cord, and Ex. 1, draw a st. line at rt. ∠s to a given st. line.

7. Show that, if the sides of a △ are represented by $m^2 + n^2$, $m^2 - n^2$, $2\,mn$, where m and n are any numbers, the △ is rt.-∠d.

Use this result to find numbers representing the sides of a rt.-∠d △.

Miscellaneous Exercises (1)

(Use a protractor in this Exercise.)

1. Draw a ||gm with diagonals 2 inches and 4 inches and their $\angle$ of intersection 50°.

2. Draw a ||gm with diagonals 4 inches and 7 inches and one side 5 inches.

3. Draw a ||gm with side 3 inches, diagonal $2\frac{1}{2}$ inches and $\angle$ 35°. Show that there are two solutions.

4. Draw a ||gm with side $2\frac{3}{8}$ inches, $\angle$ 70° and diagonal opposite $\angle$ of 70° equal to 4 inches.

5. Draw a rectangle having the perimeter 8 inches and an $\angle$ between the diagonals 80°.

6. Draw a rectangle having the difference of two sides 1 inch and an $\angle$ between the diagonals 50°.

7. Draw a rectangle which has the perimeter 9 inches and a diagonal $3\frac{1}{2}$ inches.

8. Draw an $\angle$ of 55°. Find within the $\angle$ a point which is 1 inch from one arm and 2 inches from the other.

9. Construct a $\triangle$ in which side $a = 7$ cm., $b + c = 10{\cdot}6$ cm. and $\angle \mathsf{A} = 78°$.

10. Construct a $\triangle$ with perimeter 4 inches and $\angle$s 70° and 50°.

11. **AB**, **CD** are two || st. lines; **P**, **Q** two fixed points. Find a point equidistant from **AB**, **CD** and also equidistant from **P** and **Q**. When is this impossible?

Miscellaneous Exercises (2)

1. If a quadrilateral be bisected by each of its diagonals, it is a ||gm.

2. If any point P in the diagonal AC of the ||gm ABCD be joined to B and D, the ||gm is divided into two pairs of equal $\triangle$s.

3. The diagonals of a ||gm divide the ||gm into four equal parts.

4. If two sides of a quadrilateral are || to each other, the st. line joining their middle points bisects the area of the quadrilateral.

5. If two sides of a quadrilateral are || to each other, the st. line joining their middle points passes through the intersection of the diagonals.

6. If P be any point in the side AB of ||gm ABCD, and PC, PD are joined,

$$\triangle \text{PAD} + \triangle \text{PBC} = \triangle \text{PDC}.$$

7. Prove that the following method of bisecting a quadrilateral by a st. line drawn through one of its vertices is correct:—Let ABCD be the quadrilateral. Join AC, BD. Bisect BD at E. Through E draw EF || AC and meeting BC, or CD, at F. Join AF. AF bisects the quadrilateral.

Note.—*Join* AE, *and* EC.

8. If the diagonals of ||gm ABCD cut at O, and P is any point within the $\triangle$ AOB, $\triangle$ CPD = $\triangle$ APB + $\triangle$ APC + $\triangle$ BPD.

Note.—*Join* PO.

9. ABC is an isosceles $\triangle$ having AB = AC, and D is a point in the base BC, or BC produced. Prove that the difference between the squares on AD and AC = rect. BD.DC.

10. P, Q, R, S are respectively the middle points of the sides AB, BC, CD, DA in the quadrilateral ABCD. Prove that $AB^2 + CD^2 + 2\,PR^2 = CB^2 + DA^2 + 2\,QS^2$.

11. $BY \perp AC$ and $CZ \perp AB$ in $\triangle$ ABC. Prove that $BC^2 =$ rect. AB.BZ + rect. AC.CY.

12. L, M, N are three given points, and PQ a given st. line. Construct a rhombus ABCD, having its angular points A, C lying on the line PQ, and its three sides AB, BC, CD (produced if necessary) passing through L, M, N respectively.

13. Through D the middle point of the side BC of $\triangle$ ABC a st. line XDY is drawn cutting AB at X and AC produced through C at Y. Prove $\triangle$ AXY $>$ $\triangle$ ABC.

14. From the vertex A of $\triangle$ ABC draw a st. line terminated in BC and equal to the average of AB and AC.

15. AB and CD are two equal st. lines that are not in the same st. line. Find a point P such that $\triangle$ PAB $\equiv$ $\triangle$ PCD.

Show that, in general, two such points may be found.

16. EF drawn ‖ to the diagonal AC of ‖gm ABCD meets AD, DC, or those sides produced, in E, F respectively. Prove that $\triangle$ ABE $=$ $\triangle$ BCF.

17. Construct a rect. equal to a given square and such that one side equals a given st. line.

18. Find a point in one of two given intersecting st. lines such that the perpendiculars drawn from it to both the given lines may cut off from the other a segment of given length.

19. In the diagram of II—18, if BD, BE and DE be drawn, ‖gm FK − ‖gm HG $= 2\,\triangle$ EBD.

20. ABC is an isosceles $\triangle$ in which C is a rt. $\angle$, and the bisector of $\angle$ A meets BC at D. Prove that CD $=$ AB − AC.

21. Place a st. line of given length between two given st. lines so as to be $\parallel$ a given st. line.

22. Describe a $\triangle$ = a given $\parallel$gm and such that its base = a given st. line, and one $\angle$ at the base = a given $\angle$.

23. Construct a $\parallel$gm equal and equiangular to a given $\parallel$gm and such that one side is equal to a given st. line.

24. Construct a $\parallel$gm equal and equiangular to a given $\parallel$gm and such that its altitude is equal to a given st. line.

25. ABCD is a quadrilateral. On BC as base construct a $\parallel$gm equal in area to ABCD and having one side along BA.

26. Squares ABDE, ACFG have a common $\angle$ A, and A, B, C are in the same st. line. AH is drawn $\perp$ BG and produced to cut CE at K. Prove that EK = KC.

27. Make a rhombus ABCD in which $\angle A = 100°$. A circle described with centre A and radius AB cuts BC, CD at E, F respectively. Prove that AEF is an equilateral $\triangle$.

28. A st. line AB is bisected at C and divided into two unequal parts at D. Prove that $AD^2 + DB^2 = 2\,AD.DB + 4\,CD^2$.

29. ABCD is a quadrilateral in which AB $\parallel$ CD. Prove that

$$AC^2 + BD^2 = AD^2 + BC^2 + 2\,AB.CD.$$

30. Trisect a given $\parallel$gm by st. lines drawn through one of its angular points.

31. The base BC of the $\triangle$ ABC is trisected at D, E. Prove that

$$AB^2 + AC^2 = AD^2 + AE^2 + 4\,DE^2.$$

32. ACB, ADB are two rt.-$\angle$d $\triangle$s on the same side of the same hypotenuse AB, and AX, BY are $\perp$ CD produced. Prove that

$$XC^2 + CY^2 = XD^2 + DY^2.$$

33. ABC is an isosceles $\triangle$, and XY is $\parallel$ BC and terminated in AB, AC. Prove

$$BY^2 = CY^2 + BC.XY.$$

34. Any rect. = half the rect. contained by the diagonals of the squares on two of its adjacent sides.

35. ABCD is a $\parallel$gm in which BD = AB. Prove that $BD^2 + 2\,BC^2 = AC^2$.

36. A rect. BDEC is described on the side BC of a $\triangle$ ABC. Prove that

$$AB^2 + AE^2 = AC^2 + AD^2.$$

37. BE, CD are squares described externally on the sides AB, AC of a $\triangle$ ABC. Prove that

$$BC^2 + ED^2 = 2\,(AB^2 + AC^2).$$

NOTE.—*Draw* EX, CY $\perp$ DA, AB *respectively, and rotate* $\triangle$ ABC *to the position in which* AB *coincides with* AE.

38. ABC is a $\triangle$ in which AX $\perp$ BC, and D is the middle point of BC. Prove that the difference of the squares on AB, AC = 2 BC.DX.

39. BC is the greatest and AB the least side in $\triangle$ ABC. D, E, F are the middle points of BC, CA, AB respectively; and X, Y, Z are the feet of the $\perp$s from A, B, C to the opposite sides. Prove that CA.EY = AB.FZ + BC.DX.

40. ABCD is a rect. in which E is any point in BC and F is any point in CD. Prove that ABCD = 2 $\triangle$ AEF + BE.DF.

41. A and B are two fixed points. Find the position of a point P such that $PA^2 + PB^2$ may be the least possible.

42. From a given point A draw three st. lines AB, AC, AD respectively equal to three given st. lines, and such that B, C, D are in the same st. line and BC = CD.

43. Find the locus of a point such that the sum of the squares on its distances from two given points is constant.

44. Find the locus of a point such that the difference of the squares on its distances from two given points is constant.

45. ABCD is a ||gm, P any point in BC, and Q any point in AP. Prove that △ BQC = △ PQD.

46. ABCD is a quadrilateral having AB ∥ CD, and AB + CD = BC. Prove that the bisectors of ∠s B and C intersect on AD.

47. ABC is a △ in which ∠ A is a rt. ∠, and AB > AC. Squares BCDE, CAHF, ABGK are described outwardly to the △. Prove that

$$DG^2 - EF^2 = 3\,(AB^2 - AC^2).$$

48. In the hypotenuse AB of a rt.-∠d △ ACB, points D and E are taken such that AD = AC and BE = BC. Prove that

$$DE^2 = 2\,BD\,.\,AE.$$

49. A st. line is 8 cm. in length. Divide it into two parts such that the difference of the squares on the parts = 5 sq. cm.

50. A and B are two given points and CD is a given st. line. Find a point P in CD such that the difference of the squares on PA and PB may be equal to a given rectangle.

51. AD is a median of the acute-∠d △ ABC; DX ⊥ AB, DY ⊥ AC. Prove that

$$BA\,.\,AX + CA.AY = 2\,AD^2.$$

52. Find a point P within a given quadrilateral KLMN such that △ PLM = △ PMN = △ PNK.

53. ABC is an isosceles △ in which AB = AC. AP ∥ BC. Prove that the difference between PB^2 and PC^2 equals 2 AP.BC.

54. If the sum of the squares on the diagonals of a quadrilateral be equal to the sum of the squares on the sides, the quadrilateral is a ||gm.

55. **D** is a point in the side **BC** of a $\triangle$ **ABC** such that $AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2$. $AX \perp BC$. Prove that either $BD = DC$, or $2\,DX = BC$.

56. **ABCD** is a ||gm, and **P** is a point such that $PA^2 + PC^2 = PB^2 + PD^2$. Prove that **ABDC** is a rectangle.

57. **A, B, C, D** are four fixed points. Find the locus of a point **P** such that $PA^2 + PB^2 + PC^2 + PD^2$ is constant.

58. **A, B, C, D** are four fixed points. Find the locus of a point **P** such that $PA^2 + PB^2 = PC^2 + PD^2$.

59. **D** and **E** are taken in the base **BC** of $\triangle$ **ABC** so that $BD = EC$. Through **D, E** st. lines are drawn || **AB** and **AC** forming two ||gms with **AD, AE** as diagonals. Prove the ||gms equal in area.

60. A st. line **EF** drawn || to the diagonal **AC** of a ||gm **ABCD** meets **AB** in **E** and **BC** in **F**. Prove that **BD** bisects the quadrilateral **DEBF**.

61. **ABC** is an isosceles rt.-$\angle$d $\triangle$ in which $AB = AC$. **E** is taken in **AB** and **D** in **AC** produced such that $EB = CD$. Prove that $\triangle\,EAD < \triangle\,ABC$.

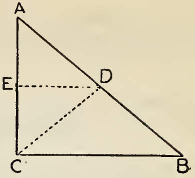
62. **L** and **M** are respectively the middle points of the diagonals **BD** and **AC** of a quadrilateral **ABCD**. **ML** is produced to meet **AD** at **E**. Prove that $\triangle$ **EBC** = half the quadrilateral.

63. **DE** is || **BC** the base of $\triangle$ **ABC**, and meets **AB, AC** at **D, E** respectively. **DE** is produced to **F** making $DF = BC$. Prove that $\triangle\,AEF = \triangle\,BDE$.

64. Inscribe a rhombus in a given ||gm, such that one vertex of the rhombus is at a given point in a side of the ||gm.

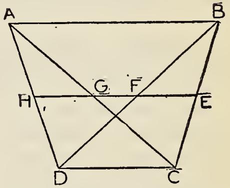
65. **ABC** is an isosceles $\triangle$ in which $AB = AC$. From **P**, any point in **BC**, **PX, PY** are drawn $\perp$ **AB, AC** respectively and **BM** is $\perp$ **AC**. Prove that $PX + PY = BM$.

If **P** is taken on **CB** produced, prove that $PY - PX = BM$.



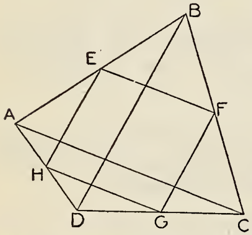
66. The middle point of the hypotenuse of a rt.-$\angle$d $\triangle$ is equidistant from the three vertices.

Note.—*Through* **D**, *the middle point of the hypotenuse* **AB**, *draw* **DE** $\parallel$ **BC**. *Join* **DC**.



67. **ABCD** is a quadrilateral in which **AB** $\parallel$ **CD**. **E, F, G, H** are the middle points of **BC, BD, AC, AD**. Prove that: (1) the st. line through **E** $\parallel$ **AB**, or **DC**, passes through **F, G** and **H**; (2) **HE** = half the sum of **AB** and **DC**; (3) **GF** = half the difference of **AB** and **DC**.

68. **E, F, G, H** are the middle points of the sides **AB, BC, CD, DA** of the quadrilateral **ABCD**. Prove that **EFGH** is a $\parallel$gm. Show also that: (1) the perimeter of **EFGH** = **AC** + **BD**; (2) if **AC** = **BD**, **EFGH** is a rhombus; (3) if **AC** $\perp$ **BD**, **EFGH** is a rectangle; (4) if **AC** = and $\perp$ **BD**, **EFGH** is a square.



69. The middle points of a pair of opposite sides of a quadrilateral and the middle points of the diagonals are the vertices of a $\parallel$gm.

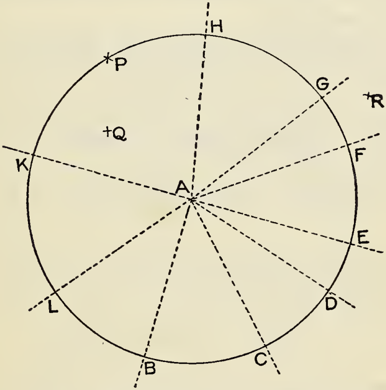
70. The st. lines joining the middle points of the opposite sides of a quadrilateral and the st. line joining the middle points of the diagonals are concurrent.

BOOK III.

LOCI
THE CIRCLE

Loci

98. A is a point and from A straight lines are drawn in different directions in the same plane.



On each line a distance of one inch is measured from A and the resulting points are B, C, D, etc.

Is there any one line that contains all the points in the plane that are at a distance of one inch from A?

To answer this question, describe a circle with centre A and radius one inch. The circumference of this circle is a line that passes through all the points.

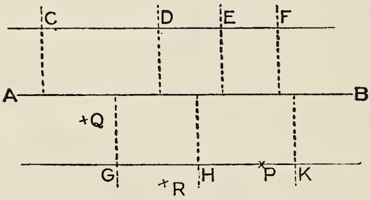
Mark any other point P on the circumference. What is the distance of P from A? From the definition of a circle the answer to this question is one inch.

If any point Q be taken within the circle, its distance from A is less than one inch, and if any point R be taken without the circle, its distance from A is greater than one inch.

Thus every point in the circumference satisfies the condition of being just one inch from A, and no point, in the plane, that is not on the circumference does satisfy this condition.

This circumference is called the **locus** of all points in the plane that are at a distance of one inch from A.

99. AB is a straight line of indefinite length, to which any number of perpendiculars are drawn.



On each of these perpendiculars a distance of one centimetre is measured from AB, and the resulting points are C, D, E, etc.

Are there any lines that contain all the points, such as C, D, etc., that are at a distance of one centimetre from AB?

Draw two straight lines parallel to AB, each at a distance of one centimetre from AB, and one or other of these lines will pass through each of the points.

Any point P in CF, or in GK, is at a distance of one centimetre from AB; any point Q in the space between CF and GK is less than one centimetre from AB, and any point R in the plane and neither between CF and GK nor in one of these lines is more than one centimetre from AB.

Thus every point in CF and GK satisfies the condition of being just one centimetre from AB, and no point outside of these lines and in the plane does satisfy this condition.

The two lines CF, GK make up the **locus** of all points in the plane that are at a distance of one centimetre from AB.

Definition.—When a figure consisting of a line or lines contains all the points that satisfy a given condition, and no others, this figure is called the locus of these points.

100. In place of speaking of the "locus of the points which satisfy a given condition," the alternative expression "locus of the point which satisfies a given condition" may be used.

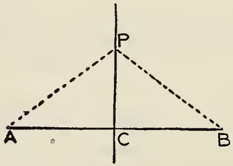
Suppose a point to move in a plane so that it traces out a continuous line, but its distance from a fixed point A in the plane is always one inch; then it must move on the circumference of the circle of centre A and radius one inch, and the locus of the point in its different positions is that circumference.

The following definition of a locus may thus be given as an alternative to that in § 99.

Definition.—**If a point moves on a line, or on lines, so that it constantly satisfies a given condition, the figure consisting of the line, or lines, is the locus of the point.**

Example 1 Theorem

The locus of a point which is equidistant from two given points is the right bisector of the straight line joining the two given points.



Hypothesis.—P is a point equidistant from A and B.

Prove that P is on the right bisector of AB.

Construction.—Bisect AB at C.

Join PC, PA, PB.

Proof.—

In $\triangle$s PAC, PBC, $\left\{\begin{array}{l} PA = PB, \\ AC = CB, \\ PC \text{ is common,} \end{array}\right.$

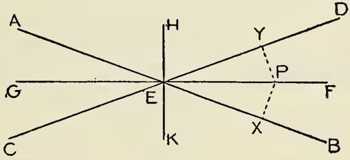
$\therefore \triangle PAC \equiv \triangle PCB,$ (I—3.)

$\therefore \angle PCA = \angle PCB,$

and $\therefore$ P is on the right bisector of AB.

EXAMPLE 2 THEOREM

The locus of a point which is equidistant from two given intersecting straight lines is the pair of straight lines which bisect the angles between the two given straight lines.



Hypothesis.—P is a point equidistant from AB and CD.

Prove that P is on the bisector of one of the $\angle$s made by AB, CD, and that all points on these bisectors are equidistant from AB, CD.

Construction.—Take P within the $\angle$ BED. Draw PX $\perp$ AB, PY $\perp$ CD and join PE.

Proof.—In $\triangle$s PEX, PEY, $\begin{cases} PX = PY, \\ PE \text{ is common,} \\ \angle PXE = \angle PYE. \end{cases}$

$\therefore \angle PEX = \angle PEY;$ (I—13 Cor.)

and $\therefore$ P is on the bisector of $\angle$ BED.

In the same manner it may be shown that, if P is within one of the other three $\angle$s made by AB, CD, it is on the bisector of that $\angle$.

Also it is easily seen, by (I—4) that all points in the bisectors are equidistant from AB and CD; and, by (I—1) and § 33, that GEF and HEK are st. lines.

$\therefore$ the locus of points equidistant from AB, CD consists of GEF and HEK.

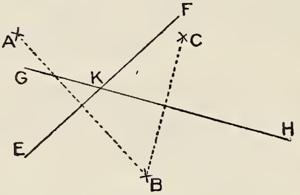
Example 3 Problem

Problem.—Find the point that is equally distant from three given points which are not in the same straight line.

Let A, B, C be the three given points.

It is required to find a point equally distant from A, B and C.

Draw EF the locus of all points that are equally distant from A and B. (Loci Ex. 1.)



Draw GH the locus of all points that are equally distant from B and C.

Let EF and GH meet at K.

Then K is the required point.

K is on EF, ∴ KA = KB.

K is on GH, ∴ KB = KC.

Consequently K is equally distant from A, B and C.

101.—Exercises

1. Find the locus of the centres of all circles that pass through two given points.

2. Describe a circle to pass through two given points and have its centre in a given st. line.

3. Describe a circle to pass through two given points and have its radius equal to a given st. line. Show that generally two such circles may be described. When will there be only one? and when none?

4. Find the locus of a point which is equidistant from two given || st. lines.

5. In a given st. line find two points each of which is equally distant from two given intersecting st. lines.

When will there be only one solution?

6. Find the locus of the vertices of all $\triangle$s on a given base which have the medians drawn to the base equal to a given st. line.

7. Find the locus of the vertices of all $\triangle$s on a given base which have one side equal to a given st. line.

8. Construct a $\triangle$ having given the base, the median drawn to the base, and the length of one side.

9. Find the locus of the vertices of all $\triangle$s on a given base which have a given altitude.

10. Construct a $\triangle$ having given the base, the median drawn to the base, and the altitude.

11. Construct a $\triangle$ having given the base, the altitude and one side.

12. Show that, if the ends of a st. line of constant length slide along two st. lines at rt. $\angle$s to each other, the locus of its middle point is a circle.

13. **AB** is a st. line and **C** is a point at a distance of 2 cm. from **AB**. Find a point which is 1 cm. from **AB** and 4 cm. from **C**. How many such points can be found?

14. Two st. lines, **AB**, **CD**, intersect each other at an $\angle$ of 45°. Find all the points that are 3 cm. from **AB** and 2 cm. from **CD**.

15. **ABC** is a scalene $\triangle$. Find a point equidistant from **AB** and **AC**, and also equidistant from **B** and **C**.

16. Find a point equidistant from the three vertices of a given $\triangle$.

17. Find four points each of which is equidistant from the three sides of a $\triangle$.

NOTE.—*Produce each side in both directions.*

18. Find the locus of a point at which two equal segments of a st. line subtend equal $\angle$s.

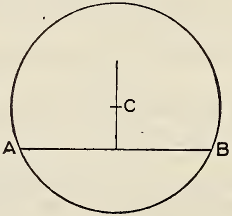
19. Find the locus of the centre of a circle which shall pass through a given point and have its radius equal to a given st. line.

The Circle

102. A definition of a circle was given in § 43, and from the explanation given in § 98 we may take the following alternative definition of it:—

A **circle** is the locus of the points in a plane that lie at a fixed distance from a fixed point.

103. As the centre of a circle is a point equally distant from the two ends of any chord of the circle, the three following statements follow at once from Loci, Ex. 1.



(*a*) **The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.**

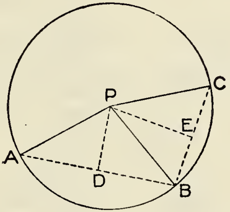
(*b*) **The straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to the chord.**

(*c*) **The right bisector of a chord of a circle passes through the centre of the circle.**

As an exercise, the pupil should give independent proofs of theorems (*a*), (*b*) and (*c*).

PROPOSITION 1 THEOREM

If from a point within a circle more than two equal straight lines are drawn to the circumference, that point is the centre.



Hypothesis.—P is a point within the circle ABC such that PA = PB = PC.

Prove that P is the centre of the circle.

Construction.—Join AB, BC, and from P draw PD $\perp$ AB and PE $\perp$ BC.

Proof.—$\because$ PA = PB,

$\therefore$ $\angle$ PAB = $\angle$ PBA. (I—2.)

In $\triangle$s PAD, PBD, $\begin{cases} \angle \text{ PDA} = \angle \text{ PDB}, \\ \angle \text{ PAD} = \angle \text{ PBD}, \\ \text{PD is common.} \end{cases}$

$\therefore$ AD = DB. (I—4.)

$\therefore$, by § 103, (*c*), the centre of the circle is somewhere in PD.

In the same manner it may be shown that the centre of the circle ABC is somewhere in PE.

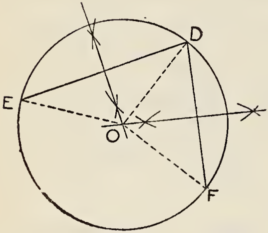
But P is the only point common to PD and PE.

$\therefore$ P is the centre of circle ABC.

Constructions

Proposition 2 Problem

Find the centre of a given circle.



Let **DEF** be the given circle.

Construction.—From any point **D** on the circumference draw two chords **DE, DF.**

Draw the right bisectors of **DE, DF** meeting at **O.**

O is the centre of circle **DEF.**

Join **OD, OE, OF.**

Proof.—The right bisector of **DE** passes through the centre. (§ 103 (*c*).)

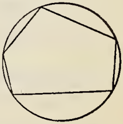
Similarly, the right bisector of **DF** passes through the centre.

But **O** is the only point common to these right bisectors.

∴ **O** is the centre of the circle.

---

104. **Definitions.**—If a circle passes through all the vertices of a rectilineal figure, it is said to be **circumscribed** about the figure.



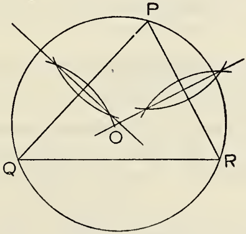
Four points so situated that a circle may be described to pass through all of them are said to be **concyclic.**

If the four vertices of a quadrilateral are on the circumference of the same circle, it is said to be a **cyclic quadrilateral.**

The centre of a circle circumscribed about a triangle is called the **circumcentre** of the triangle.

Proposition 3 Problem

Circumscribe a circle about a given triangle.



Let PQR be the given △.

Construction.—Draw the right bisectors of PQ, PR meeting at O.

∵ O is on the right bisector of PQ.

∴ OP = OQ. (Loci, Ex.1.)

Similarly OP = OR.

∴ OP = OQ = OR,

And a circle described with centre O and radius OP will pass through Q and R, and be circumscribed about the △.

105.—Exercises

1. Through a given point within a circle draw a chord that is bisected at the given point.

2. Complete a circle of which only an arc is given.

3. Circumscribe a circle about a given square.

4. Circumscribe a circle about a given rectangle.

5. Describe a circle with a given centre to cut a given circle at the ends of a diameter.

6. The locus of the middle points of a system of || chords in a circle is a diameter of the circle.

7. If two circles cut each other, the st. line joining their centres bisects their common chord at rt. $\angle$s.

8. If each of two equal st. lines has one extremity on one of two concentric circles and the other extremity on the other circle, the st. lines subtend equal $\angle$s at the common centres.

9. A st. line cuts the outer of two concentric circles at E, F; and the inner at G, H. Prove that $EG = FH$.

10. A st. line cannot cut a circle at more than two points.

11. Two chords of a circle cannot bisect each other unless both are diameters.

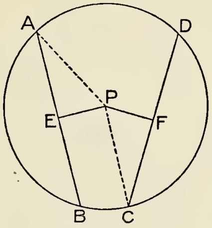
12. A circle cannot be circumscribed about a ||gm unless the ||gm is a rectangle.

13. A st. line which joins the middle points of two || chords in a circle is $\perp$ to the chords.

14. If two circles cut each other, a st. line through a point of intersection, || to the line of centres and terminated in the circumferences, is double the line joining the centres.

PROPOSITION 4 THEOREM

If two chords are equally distant from the centre of a circle, the chords are equal to each other.



Hypothesis.—ABC is a circle of which P is the centre and AB, CD are two chords such that the $\perp$s PE, PF from P to AB, CD respectively are equal to each other.

Prove that AB = CD.

Construction.—Join AP, CP.

Proof.— $\because$ $\angle$ AEP is a rt. $\angle$,

$\therefore$ $AE^2 + EP^2 = AP^2$. (II—23.)

Similarly $CF^2 + FP^2 = CP^2$.

But $\because$ $AP = CP$,

$\therefore$ $AP^2 = CP^2$.

$\therefore$ $AE^2 + EP^2 = CF^2 + FP^2$.

But $\because$ $EP = FP$,

$\therefore$ $EP^2 = FP^2$.

$\therefore$ $AE^2 = CF^2$

and $\therefore$ $AE = CF$.

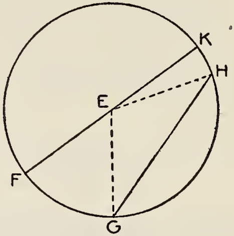
$AB = 2\,AE$, (§ 103, (*a*).)

and $CD = 2\,CF$.

$\therefore$ $AB = CD$.

PROPOSITION 5 THEOREM

If a chord of a circle does not pass through the centre, the chord is less than a diameter.



Hypothesis.—In the circle FGH, GH is a chord which does not pass through the centre and FK is a diameter. E is the centre.

Prove that GH $<$ FK.

Construction.—Join EG, EH.

Proof.— $\because$ GE $=$ EF and EH $=$ EK,

$\therefore$ GE $+$ EH $=$ FK.

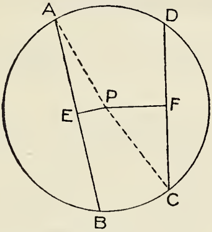
$\because$ GEH is a $\triangle$,

$\therefore$ GH $<$ GE $+$ EH.

And $\therefore$ GH $<$ FK.

PROPOSITION 6 THEOREM

Of two chords in a circle the one which is nearer to the centre is greater than the one which is more remote from the centre.



Hypothesis.—P is the centre of a circle ABC, and AB, CD are two chords such that PE, the distance of AB from the centre, is less than PF, the distance of CD from the centre.

Prove that AB > CD.

Construction.—Join PA, PC.

Proof.— $\because$ PEA is a rt. $\angle$.

$\therefore AE^2 + EP^2 = AP^2.$ (II—23.)

Similarly $CF^2 + FP^2 = CP^2.$

But $\because AP = CP,$

$\therefore AP^2 = CP^2.$

And $\therefore AE^2 + EP^2 = CF^2 + FP^2.$

$\because EP < PF,$

$\therefore EP^2 < PF^2.$

And $\therefore AE^2 > CF^2,$

$\therefore AE > CF.$

But $AB = 2\,AE,$

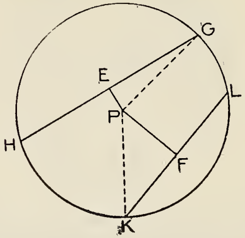
and $CD = 2\,CF,$

$\therefore AB > CD.$

PROPOSITION 7 THEOREM

(Converse of III—6)

If two chords of a circle are unequal, the greater is nearer to the centre than the less.



Hypothesis.—Chord GH > chord KL, and PE, PF are respectively perpendiculars from the centre P to GH, KL.

Prove that PE < PF.

Construction.—Join PG, PK.

Proof.— $\because$ PEG is a rt. $\angle$,

$GE^2 + EP^2 = GP^2$. (II—23.)

Similarly, $PF^2 + FK^2 = PK^2$.

$\therefore GE^2 + EP^2 = PF^2 + FK^2$.

But $\because GH = 2\,GE$ and $KL = 2\,KF$,

and also $GH > KL$,

$\therefore GE > KF$.

$\therefore GE^2 > KF^2$.

Hence, $EP^2 < PF^2$.

And $\therefore EP < PF$.

106.—Exercises

1. If two chords of a circle are equal to each other, they are equally distant from the centre. (Converse of III—4.)

2. A chord 6 cm. in length is placed in a circle of radius 4 cm. Calculate the distance of the chord from the centre.

3. A chord a inches long is placed in a circle of radius b inches. Find an algebraic expression for the distance of the chord from the centre.

4. In a circle of radius 5 cm. a chord is placed at a distance of 3 cm. from the centre. Calculate the length of the chord.

5. Through a given point within a circle draw the shortest chord.

6. In a circle of radius 4 cm., a point **P** is taken at the distance 3 cm. from the centre. Calculate the length of the shortest chord through **P**.

7. The length of a chord 2 cm. from the centre of a circle is 5·5 cm. Find the length of a chord 3 cm. from the centre. Verify your result by measurement.

8. In a circle of radius 5 cm., two || chords of lengths 8 cm. and 6 cm. are placed. Find the distance between the chords. Show that there are two solutions.

9. **ACB** is a diameter, and **C** the centre of a circle. **D** is any point on **AB**, or on **AB** produced, and **P** is any point on the circumference except **A** and **B**. Show that **DP** is intermediate in magnitude between **DA** and **DB**.

10. **O** is the centre of a circle, and **P** is any point. If two st. lines be drawn through **P**, cutting the circle and making equal $\angle$s with **PO**, the chords intercepted on these lines by the circumference are equal to each other.

11. **O** is the centre of a circle, and **P** is any point. On two lines drawn through **P** chords **AB**, **CD** are intercepted by the circumference. If the $\angle$ made by **AB** with **PO** $>$ $\angle$ made by **CD** with **PO**, the chord **AB** $<$ chord **CD**.

12. From any point in a circle which is not the centre equal st. lines can be drawn to the circumference only in pairs.

13. Find the locus of the middle points of chords of a fixed length in a circle.

14. K and L are two fixed points. Find a point P on a given circle such that $KP^2 + LP^2$ may be the least possible.

15. Chords equally distant from the centre of a circle subtend equal $\angle$s at the centre.

16. The nearer to the centre of two chords of a circle subtends the greater $\angle$ at the centre.

Answers:—2, 26·5 mm. nearly; 4, 8 cm.; 6, 5·3 cm. nearly; 7, 32 mm. nearly; 8, 1 cm. or 7 cm.

Angles in a Circle

Proposition 8 Theorem

The angle which an arc of a circle subtends at the centre is double the angle which it subtends at any point on the remaining part of the circumference.

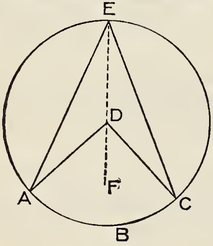


Fig. 1

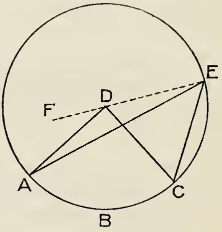


Fig. 2

Hypothesis.—ABC is an arc of a circle, D the centre, and E any point on the remaining part of the circumference.

Prove that $\angle$ ADC = 2 $\angle$ AEC.

Construction.—Join ED and produce ED to any point F.

Proof.—

In both figures:—

In $\triangle$ DAE, $\because$ DA = DE,

$\therefore \angle$ DAE = $\angle$ DEA (I—2.)

$\because$ ADF is an exterior $\angle$ of $\triangle$ ADE,

$\therefore \angle$ ADF = $\angle$ DAE + $\angle$ DEA

= 2 $\angle$ DEA.

Similarly $\angle$ CDF = 2 $\angle$ DEC.

In Fig. 1:—

$\angle$ ADF = 2 $\angle$ DEA

$\angle$ CDF = 2 $\angle$ DEC,

adding, $\angle$ ADC = 2($\angle$ DEA + $\angle$ DEC)

= 2 $\angle$ AEC.

In Fig. 2:—

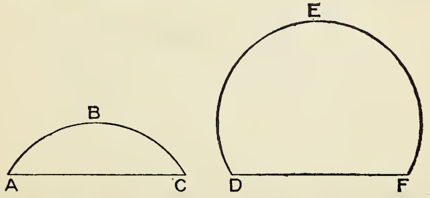
$\angle$ CDF = 2 $\angle$ DEC,

$\angle$ ADF = 2 $\angle$ DEA,

subtracting, $\angle$ ADC = 2($\angle$ DEC − $\angle$ DEA).

= 2 $\angle$ AEC.

107. **Definitions.**—The figure bounded by an arc of a circle and the chord which joins the ends of the arc is called a **segment of a circle.**



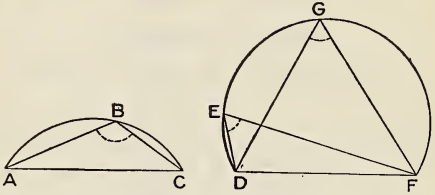
ABC, DEF are segments of circles.

A semi-circle is a particular case of a segment.

An arc is called a **major arc** or a **minor arc** according as it is greater or less than half the circumference.

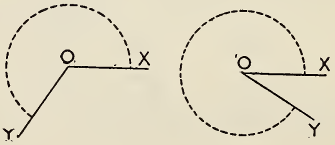
A segment is called a **major segment** or a **minor segment** according as the arc of the segment is a major or a minor arc.

108. **Definitions.**—If the ends of a chord of a segment are joined to any point on the arc of the segment, the angle between the joining lines is called an **angle in the segment.**



ABC is an $\angle$ **in** the segment **ABC**, and **DEF** is an $\angle$ **in** the segment **DEF**. **DGF** is also an $\angle$ **in** the segment **DEF**.

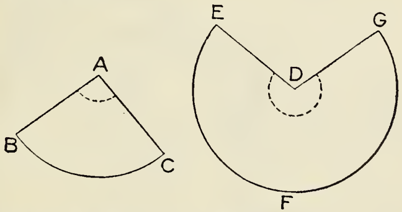
109. **Definitions.**—An angle which is greater than two right angles but less than four right angles is called a **reflex** angle.



A straight line starting from the position **OX** and rotating in the direction opposite to that of the hands

of a clock to the position **OY**, in either diagram, traces out the reflex angle **XOY**.

The figures bounded by two radii of a circle and either of the arcs intercepted by the radii is called a **sector of the circle.**

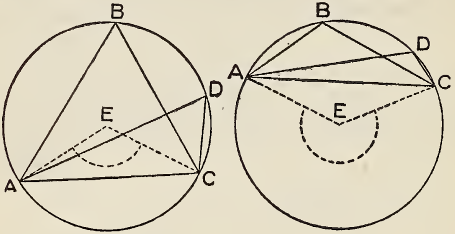


ABC, DEFG are sectors of circles.

BAC is the ∠ of the sector **ABC**, and the reflex ∠ **EDG** is the ∠ of the sector **DEFG.**

Proposition 9 Theorem

If angles are in the same segment of a circle, the angles are equal to each other.



Hypothesis.—ABC, ADC are two ∠s in the same segment ABDC.

Prove that ∠ ABC = ∠ ADC.

Construction.—Find E the centre of the circle. Join AE, EC.

Proof.—The ∠ AEC at the centre and the ∠s ABC and ADC at the circumference are subtended by the same arc,

$$\therefore \angle \text{ ABC} = \tfrac{1}{2} \angle \text{ AEC}, \qquad \text{(III—8.)}$$

$$\text{and } \angle \text{ ADC} = \tfrac{1}{2} \angle \text{ AEC},$$

$$\therefore \angle \text{ ABC} = \angle \text{ ADC}.$$

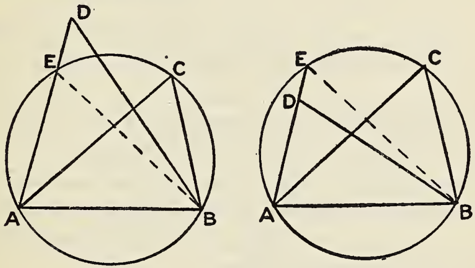
Alternative statement of the preceding theorem:—

The angle in a given segment is constant in magnitude for all positions of the vertex of the angle on the arc of the segment.

Proposition 10 Theorem

(Converse of III—9)

If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.



Hypothesis.—Let A, B, C, D be four points such that the st. line AB subtends at C and D on the same side of AB equal $\angle$s ACB, ADB.

Prove that A, B, C and D are concyclic.

Proof.—Draw a circle through A, B, C and, if possible, let this circle not pass through D.

Let AD, or AD produced, meet the circle at E.

Join BE.

$\angle$s AEB, ACB are in the same segment.

$\therefore\ \angle$ AEB $=$ $\angle$ ACB. (III—9.)

But $\angle$ ADB $=$ $\angle$ ACB. (*Hyp.*)

$\therefore\ \angle$ AEB $=$ $\angle$ ADB.

i.e., the exterior $\angle$ of a $\triangle$ $=$ the interior, non-adjacent $\angle$, which is impossible.

$\therefore$ the circle through A, B and C passes through D.

Cor. The locus of the vertex of a △ drawn on one side of a given base and having a given vertical ∠ is the arc of a segment on that base as chord.

110. **Definition.**—If the three angles of one triangle are respectively equal to the three angles of another triangle, the triangles are said to be **similar.**

111. There are two conditions implied when figures are said to be similar: not only are the angles of one respectively equal to the angles of the other, but a certain relationship must exist between the lengths of the sides of the two figures. For triangles, it will be shown in Book IV that, if one of these conditions is given, the other is also true. For figures of more than three sides this is not the case, and a definition including both conditions must be given.

The symbol ||| may be used for the word similar, or for "is similar to."

112.—Exercises

1. Prove III—8 when the arc is half the circumference.

2. Construct a circular arc on a chord of 3 inches and having the apex 3 inches from the chord. Calculate the radius of the circle.

3. If the chord of an arc be a inches, and the distance of its apex from the chord b inches, show that the radius of the circle is $\dfrac{a^2 + 4b^2}{8b}$.

4. Two chords **AOB, COD,** intersect at a point **O** within the circle. Show that **AOC, BOD** are similar △s. **BOC, AOD** are also similar △s. Read the segments that contain the equal ∠s.

5. **ABC** is a $\triangle$ inscribed in a circle, and the bisector of $\angle$ **A** meets the circumference again at **D**. Show that the st. line drawn from **D** $\perp$ **BC** is a diameter.

6. A circle is divided into two segments by a chord equal to the radius. Show that the $\angle$ in the major segment is 30° and that in the minor segment is 150°.

7. The locus of the vertices of the rt. $\angle$s of all rt.-$\angle$d $\triangle$s on the same hypotenuse is a circle.

8. Prove III—8 when the arc is greater than half the circumference.

9. **PQR** is a $\triangle$ inscribed in a circle. The bisector of $\angle$ **P** cuts **QR** at **D** and meets the circle at **E**. Prove that $\triangle$ **PQD** $|||$ $\triangle$ **PER**.

10. **DPQ** and **EPQ** are two fixed circles, and **D, P** and **E** are in the same st. line. The bisector of $\angle$ **DQE** meets **DE** at **F**. Show that the locus of **F** is an arc of a circle.

11. If the diagonals of a quadrilateral inscribed in a circle cut at rt. $\angle$s, the $\perp$ from their intersection on any side bisects the opposite side.

12. If the diagonals of a quadrilateral inscribed in a circle cut at rt. $\angle$s, the distance of the centre of the circle from any side is half the opposite side.

13. If the diagonals of a quadrilateral inscribed in a circle cut each other at rt. $\angle$s, the $\angle$s which a pair of opposite sides of the quadrilateral subtend at the centre of the circle are supplementary.

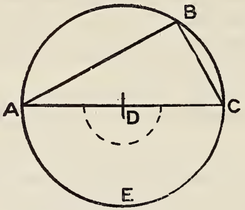
14. **XYZ, XYV** are two equal circles, the centre of each being on the circumference of the other. **ZXV** is a st. line. Prove that **YZV** is an equilateral $\triangle$.

15. **EFGH** is a quadrilateral inscribed in a circle and **EF** = **GH**. Prove that **EG** = **FH**.

16. **ABCD** is a quadrilateral inscribed in a circle; the diagonals **AC, BD** cut at **E**; **F** the centre of the circle is within the quadrilateral. Prove that $\angle$ **AFB** + $\angle$ **CFD** = 2 $\angle$ **AEB**.

Proposition 11 Theorem

The angle in a semi-circle is a right angle.



Hypothesis.—ABC is an $\angle$ in the semi-circle ABC, of which D is the centre.

Prove that ABC is a rt. $\angle$.

Proof.—The $\angle$ ABC at the circumference, and the st. $\angle$ ADC at the centre, subtend the same arc AEC.

$$\therefore \angle ABC = \tfrac{1}{2} \angle ADC. \qquad \text{(III—8.)}$$

$$= \text{a rt. } \angle.$$

Proposition 12 Theorem

(*a*) **The angle in a major segment of a circle is acute.**

(*b*) **The angle in a minor segment of a circle is obtuse.**

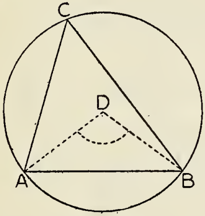


Fig 1

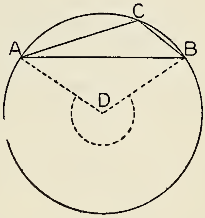


Fig.2

(*a*) *Hypothesis.*—ACB is an ∠ in a major segment of a circle. (Fig. 1.)

Prove that ∠ ACB is acute.

Construction.—Join A and B to the centre D.

Proof.—∠ ACB at the circumference and ∠ ADB at the centre stand on the same arc,

$\therefore$ ∠ ACB = $\frac{1}{2}$ ∠ ADB. (III—8.)

But ∠ ADB is $<$ a st. ∠.

$\therefore$ ∠ ACB is acute.

(*b*) *Hypothesis.*—ACB is an ∠ in a minor segment of a circle. (Fig. 2.)

Prove that ∠ ACB is obtuse.

Construction.—Join A and B to the centre D.

Proof.—

∠ ACB = $\frac{1}{2}$ the reflex ∠ ADB. (III—8.)

$\therefore$ ∠ ACB is obtuse.

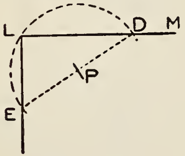
113.—Exercises

1. A circle described on the hypotenuse of a rt.-∠d △ as diameter passes through the vertex of the rt. ∠. (Converse of III—11.)

2. Circles described on two sides of a △ as diameters intersect on the third side or the third side produced.

Where is the point of intersection when the circles are described on the equal sides of an isosceles △?

3. **LM** is a st. line and **L** a point from which it is required to draw a ⊥ to **LM**.



Construction.—With a convenient point **P** as centre describe a circle to pass through **L** and cut **LM** at **D**. Join **DP**, and produce **DP** to cut the circle at **E**. Join **LE**.

Prove **LE** ⊥ **LM**.

4. **EF**, **EG** are diameters of two circles **FEH**, **GEH** respectively. Show that **FHG** is a st. line.

5. **ST** is a diameter of the circle **SVT**. A circle is described with centre **S** and radius **ST**. Show that any chord of this latter circle drawn from **T** is bisected by the circle **SVT**.

6. Chords of a given circle are drawn through a given point. Find the locus of the middle points of the chords when the given point is (*a*) on the circumference, (*b*) within the circle, (*c*) without the circle.

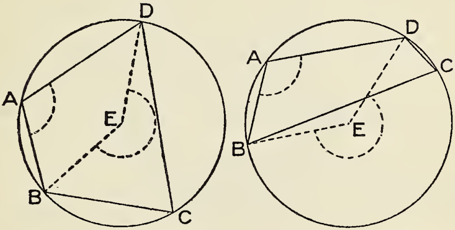
7. **F** is any point on the arc of a semi-circle of which **DE** is a diameter. The bisectors of ∠s **FED**, **FDE** meet at **P**. Find the locus of **P**.

8. **F** is a point on the arc of a semi-circle of which **DE** is a diameter. **FG** ⊥ **DE**. Show that the △s **FDG**, **FEG**, **FDE** are similar.

9. **PQRS** is a st. line and circles described on **PR**, **QS** as diameters cut at **E**. Prove that ∠ **PEQ** = ∠ **RES**.

PROPOSITION 13 THEOREM

If a quadrilateral be inscribed in a circle, its opposite angles are supplementary.



Hypothesis.—ABCD is a quadrilateral inscribed in a circle.

Prove that $\angle$ A + $\angle$ C = 2 rt. $\angle$s.

Construction.—Find the centre E. Join BE, ED.

Proof.—$\angle$ BED at the centre and $\angle$ C at the circumference are subtended by the same arc BAD.

$\therefore$ $\angle$ C = $\frac{1}{2}$ $\angle$ BED. (III—8.)

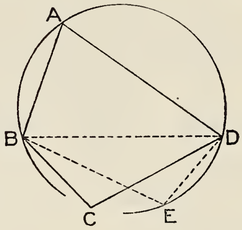
Similarly $\angle$ A = $\frac{1}{2}$ reflex $\angle$ BED.

Hence $\angle$ A + $\angle$ C = $\frac{1}{2}$ the sum of the two $\angle$s BED at the centre = $\frac{1}{2}$ of 4 rt. $\angle$s

= 2 rt. $\angle$s.

Proposition 14 Theorem

If the opposite angles of a quadrilateral are supplementary, its vertices are concyclic.



Hypothesis.—**ABCD** is a quadrilateral in which $\angle$ **A** + $\angle$ **C** = 2 rt. $\angle$s.

Prove that **A, B, C, D** are on the circumference of a circle.

Construction.—Draw a circle through the three points **A, B, D.** On this circumference and on the side of **BD** remote from **A** take a point **E.** Join **BE, ED.**

Proof.—$\because$ **ABED** is a quadrilateral inscribed in a circle,

$\therefore$ $\angle$ **A** + $\angle$ **E** = 2 rt. $\angle$s; (III—13.)

but $\angle$ **A** + $\angle$ **C** = 2 rt. $\angle$s. (*Hyp.*)

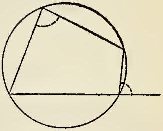
$\therefore$ $\angle$ **A** + $\angle$ **E** = $\angle$ **A** + $\angle$ **C**,

and $\therefore$ $\angle$ **E** = $\angle$ **C**.

Consequently, as **C, E** are on the same side of **BD**, the circle **BADE** passes through **C**. (III—10.)

114.—Exercises

1. If one side of an inscribed quadrilateral be produced, the exterior ∠ thus formed at one vertex equals the interior ∠ at the opposite vertex of the quadrilateral.



State and prove the converse.

2. From a point **O** without a circle two st. lines **OAB, OCD** are drawn cutting the circumference at **A, B, C, D.** Show that △s **OBC, OAD** are similar, and that △s **OAC, OBD** are similar.

3. If a ||gm be inscribed in a circle, the ||gm is a rect.

4. **A, D, C, E, B** are five successive points on the circumference of a circle; and **A, B** are fixed. Show that the sum of the ∠s **ADC, CEB** is the same for all positions of **D, C, E.**

5. A circle is circumscribed about an equilateral △. Show that the ∠ in each segment outside the △ is an ∠ of 120°.

6. A scalene △ is inscribed in a circle. Show that the sum of the ∠s in the three segments outside the △ is 360°.

7. A quadrilateral is inscribed in a circle. Show that the sum of the ∠s in the four segments outside the quadrilateral is 540°.

8. **P** is a point on the diagonal **KM** of the ||gm **KLMN.** Circles are described about **PKN** and **PLM.** Show that **LN** passes through the other point of intersection of the circles.

9. A circle drawn through the middle points of the sides of a △ passes through the feet of the ⊥s from the vertices to the opposite sides.

10. If the opposite sides of a quadrilateral inscribed in a circle are produced to meet at **L** and **M**, and about the △s so formed outside the quadrilateral circles are described intersecting again at **N**, then **L, M, N** are in the same st. line.

11. In a △ **DEF, DX** ⊥ **EF** and **EY** ⊥ **DF.** Prove that ∠ **XYF** = ∠ **DEF.**

12. **PQRS, PQTV** are circles and **SPV, RQT** are st. lines. Prove that **SR** || **VT**.

13. The st. lines that bisect any $\angle$ of a quadrilateral inscribed in a circle and the opposite exterior $\angle$ meet on the circumference.

14. **XYZ** is a $\triangle$; **YD** $\perp$ **ZX**, and **DE** $\perp$ **XY**; **ZF** $\perp$ **XY** and **FG** $\perp$ **ZX**. Show that **EG** || **YZ**.

15. **EGD, FGD** are two circles with centres **H, K** respectively. **EGF** is a st. line. **EH, FK** meet at **P**. Show that **H, K, D, P** are concyclic.

16. **KL, MN** are two || chords in a circle; **KE, NF** two $\perp$ chords in the same circle. Show that **LF** $\perp$ **ME**.

17. The bisectors of the $\angle$s formed by producing the opposite sides of a quadrilateral inscribed in a circle are $\perp$ to each other.

18. **HKM, LKM** are two circles, and **HKL** is a st. line. **HM, LM** cut the circles again at **E, F** respectively, and **HF** cuts **LE** at **G**. Show that a circle may be circumscribed about **MEGF**.

19. **PQRS** is a quadrilateral and the bisectors of the $\angle$s **P, Q; Q, R; R, S; S, P** meet at four points. Show that a circle may be circumscribed about the quadrilateral thus formed.

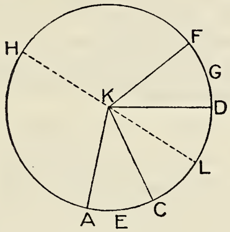
20. **EF** is the diameter of a semi-circle and **G, H** any two points on its arc. **EH, FG** cut at **K** and **EG, FH** cut at **L**. Show that **KL** $\perp$ **EF**.

21. **DE** is the diameter, **O** the centre and **P** any point on the arc of a semi-circle. **PM** $\perp$ **DE**. Show that the bisector of $\angle$ **MPO** passes through a fixed point.

22. **PQR** is a $\triangle$ and **PDQ, PFQ** are two circles cutting **PR** at **D, F** and **QR** at **E, G**. Prove that **DE** || **FG**.

PROPOSITION 15 THEOREM

If two angles at the centre of a circle are equal to each other, they are subtended by equal arcs.



Hypothesis.—AKC, DKF are equal ∠s at the centre K of the circle ACD.

Prove that arc AEC equals arc DGF.

Construction.—Draw the diameter HKL bisecting ∠ CKD.

Proof.—Suppose the circle to be folded along the diameter HKL, and the semi-circle HFL will coincide throughout with the semi-circle HAL.

∵ ∠ LKD = ∠ LKC,

∴ KD falls along KC;

and ∴ D falls on C.

∵ ∠ DKF = ∠ CKA,

∴ KF falls along KA;

and ∴ F falls on A.

∴ the arc DGF coincides with the arc CEA.

∴ arc DGF = arc CEA.

115.—Exercises

1. If two arcs of a circle are equal to each other, they subtend equal $\angle$s at the centre. (Prove either by indirect demonstration or by the construction and method used in III—15.)

2. If two $\angle$s at the circumference of a circle are equal to each other, they are subtended by equal arcs.

3. If two arcs of a circle are equal to each other, they subtend equal $\angle$s at the circumference.

4. In equal circles equal $\angle$s at the centres (or circumferences) stand on equal arcs.

5. In equal circles equal arcs subtend equal $\angle$s at the centres (or circumferences).

6. If two arcs of a circle (or of equal circles) are equal, they are cut off by equal chords.

7. If two chords of a circle are equal to each other, the major and minor arcs cut off by one are respectively equal to the major and minor arcs cut off by the other.

8. If two sectors of a circle have equal $\angle$s at the centre, the sectors are congruent.

9. Bisect a given arc of a circle.

10. Parallel chords of a circle intercept equal arcs.

Show also that the converse is true.

11. If two equal circles cut each other, any st. line drawn through one of the points of intersection will meet the circles again at two points which are equally distant from the other point of intersection.

12. The bisectors of the opposite $\angle$s of a quadrilateral inscribed in a circle meet the circumference at the ends of a diameter.

13. If two $\angle$s at the centre of a circle are supplementary, the sum of the arcs on which they stand is equal to half the circumference.

14. If any number of $\angle$s are in a segment, their bisectors all pass through one point.

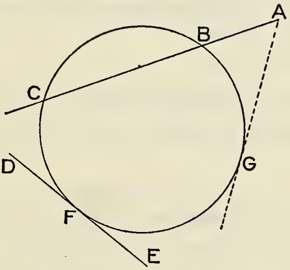
Tangents and Chords

116. Definitions.—A **secant** of a circle is a straight line intersecting the circumference in two points and extending outside the circle in one or both directions.

A straight line which, however far it may be produced, has one point on the circumference of a circle and all other points without the circle is called a **tangent** to the circle.

A tangent is said to **touch** the circle.

The common point of a tangent and circle, that is, the point where the tangent touches the circle, is called the **point of contact.**



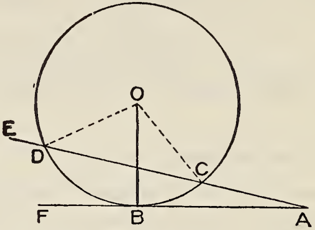
ABC is a secant drawn to the circle BCF from the point A.

DFE is a tangent to the circle BCF, touching the circle at the point of contact F.

If the secant ABC rotate about the point A until the two points B, C where it cuts the circle coincide at G, the secant becomes a tangent having G for the point of contact.

Proposition 16 Theorem

The radius drawn to the point of contact of a tangent is perpendicular to the tangent.



Hypothesis.—ABF is a tangent to the circle CBD at the point B, O is the centre and OB the radius drawn to the point of contact.

Prove that OB $\perp$ AF.

Construction.—From any point A, except B, in AF draw a secant AE cutting the circle in C and D. Join OC, OD.

Proof.— $\because$ OD $=$ OC,

$\therefore$ $\angle$ ODC $=$ $\angle$ OCD. (I—2.)

But, st. $\angle$ EDC $=$ st. $\angle$ DCA,

$\therefore$ $\angle$ ODE $=$ $\angle$ OCA.

Rotate AE about A until it coincides with AF. As AE rotates about A the $\angle$s ODE, OCA are continually equal to each other and finally $\angle$ ODE becomes $\angle$ OBF and $\angle$ OCA becomes $\angle$ OBA.

$\therefore$ $\angle$ OBF $=$ $\angle$ OBA.

and $\therefore$ OB $\perp$ AF.

Cor. 1.—Only one tangent can be drawn at any point on the circumference of a circle.

∵ only one st. line can be $\perp$ to the radius at that point.

Hence, also:—**The straight line drawn perpendicular to a radius at the point where it meets the circumference is a tangent.**

Cor. 2.—The perpendicular to a tangent at its point of contact passes through the centre of the circle.

∵ only one st. line can be $\perp$ to the tangent at that point.

Cor. 3.—The perpendicular from the centre on a tangent passes through the point of contact.

∵ only one $\perp$ can be drawn from a given external point to a given st. line.

Give proofs of these corollaries, using the indirect method.

117.—Exercises

1. Draw a tangent to a given circle from a given point on the circumference.

2. Describe a circle with its centre on a given st. line **DE** to pass through a given point **P** in **DE** and touch another given st. line **DF.**

3. Find the locus of the centres of all circles that touch a given st. line at a given point.

4. Describe a circle to pass through a given point and touch a given st. line at a given point.

5. Tangents at the ends of a diameter are $\parallel$.

6. **C** is any point on the tangent of which **A** is the point of contact. The st. line from **C** to the centre **O** cuts the circumference at **B**. **AD** is $\perp$ **OC**. Show that **BA** bisects the $\angle$ **DAC**.

7. Find the locus of the centres of all circles which touch two given $\parallel$ st. lines.

8. Draw a circle to touch two given || st. lines and pass through a given point between the ||s. Show that two such circles may be drawn.

9. To a given circle draw two tangents, each of which is || to a given st. line.

10. To a given circle draw two tangents, each of which is $\perp$ to a given st. line.

11. Give an alternative proof for III—16 by supposing the radius **OB** drawn to the point of contact of the tangent **ABF** not $\perp$ to **AF** and drawing **OG** $\perp$ **AF.**

12. Two tangents to a circle meet each other. Prove that they are equal to each other.

13. **EF** is a diameter of a circle and **EG** is a chord. **EH** is a chord bisecting the $\angle$ **FEG.** Prove that the tangent at **H** is $\perp$ **EG.**

14. Draw a circle to touch a given st. line at a given point and have its centre on another given st. line.

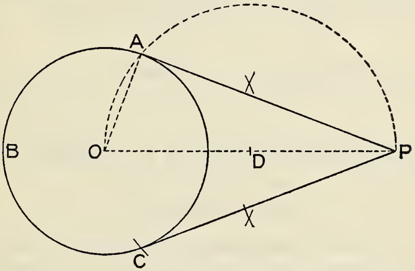
15. Draw a tangent to a given circle making a given $\angle$ with a given st. line.

Show that, in general, four such tangents may be drawn.

Construction

Proposition 17 Problem

Draw a tangent to a given circle from a given point without the circle.



Let ABC be the given circle, and P the given point.

It is required to draw a tangent from P to the circle ABC.

Join P to the centre O. Bisect OP at D. With centre D and radius DO, describe a circle cutting the circle ABC at A and C. Join PA, PC.

Either PA or PC is a tangent to the given circle.

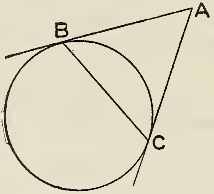
Join OA.

OAP is an ∠ in a semi-circle, and is ∴ a rt. ∠. (III—11.)

∴ PA is a tangent. (III—16, Cor. 1.)

In the same manner it may be shown that PC is a tangent.

118. **Definition.**—The straight line joining the points of contact of two tangents to a circle is called the **chord of contact** of the tangents.



BC is the chord of contact of the tangents AB, AC.

119.—Exercises

1. Draw a circle of radius 4 cm. Take a point 9 cm. from the centre of the circle. From this point draw two tangents to the circle. Measure the length of each tangent and check your result by calculation.

2. Draw a circle of radius 5 cm. Mark a point 7 cm. from the centre. From this point draw two tangents to the circle and measure the $\angle$ between the tangents. (91° nearly.)

3. Draw a circle with a radius of 3 cm. Mark any point A on the circumference, and from this point draw a tangent AB 4 cm. long. Measure the distance of B from the centre and check your result.

4. Draw a circle with 43 mm. radius. Draw any st. line through the centre, and find a point, in this line, from which the tangent to the circle will be 5 cm. in length. Measure the distance of the point from the centre and check your result.

5. Mark two points A and B 7 cm. apart. Draw two st. lines from A such that the length of the perpendicular from B to either of them is 4 cm.

6. Draw a circle of radius 6 cm. Mark a point P 4 cm. from the centre. Draw a chord through P such that the perpendicular from

the centre to the chord is 3 cm. in length. Measure the length of the chord and check your result by calculation.

7. Draw a circle of radius 36 mm. Mark any point **P** without the circle. Draw a st. line from **P** such that the chord cut off on it by the circle is 4 cm. in length.

8. Draw a circle of radius 47 mm. Mark a point **P** 4 cm. from the centre. Draw two chords through **P**, each of which is 65 mm. in length.

9. If from a point without a circle two tangents are drawn, the st. line drawn from this point to the centre bisects the chord of contact and cuts it at rt. $\angle$s.

10. If a quadrilateral be circumscribed about a circle, the sum of one pair of opposite sides equals the sum of the other pair.

11. Through a given point draw a st. line such that the chord intercepted on the line by a given circle is equal to a given st. line.

12. If a ||gm be circumscribed about a circle, the ||gm is a rhombus.

13. If two tangents to a circle are ||, their chord of contact is a diameter.

14. If two || tangents to a circle are cut by a third tangent to the circle at **A**, **B**; show that **AB** subtends a rt. $\angle$ at the centre.

15. If a quadrilateral be circumscribed about a circle, the $\angle$s subtended at the centre by a pair of opposite sides are supplementary.

16. To a given circle draw two tangents containing an $\angle$ equal to a given $\angle$.

17. Find the locus of the points from which tangents drawn to a given circle are equal to a given st. line.

18. Find a point **P** in a given st. line such that the tangent from **P** to a given circle is of given length. What is the condition on which this is possible?

19. **E** is a point outside a circle the centre of which is **D**. In **DE** produced find a point **F**, such that the length of the tangent from **F** may be twice that of the tangent from **E**.

20. Two tangents, **LM**, **LN** are drawn to a circle; **P** is any point on the circumference outside the $\triangle$ **LMN**. Prove that $\angle$ **LMP** + $\angle$ **LNP** is constant.

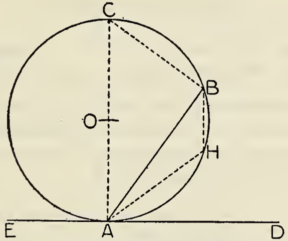
21. Find the $\angle$ between the tangents to a circle from a point whose distance from the centre is equal to a diameter.

22. Show that all equal chords of a given circle touch a fixed concentric circle.

23. From a given point without a circle draw a st. line such that the part intercepted by the circle subtends a rt. $\angle$ at the centre.

Proposition 18 Theorem

If at one end of a chord of a circle a tangent be drawn, each angle between the chord and the tangent is equal to the angle in the segment on the other side of the chord.



Hypothesis.—AB is a chord and EAD a tangent to the circle ABC.

Prove that ∠ DAB = ∠ ACB and that ∠ EAB = ∠ AHB.

Construction.—From A draw the diameter AOC. Join BC. Join any point H in the arc AHB to A and B.

Proof.—∵ ABC is an ∠ in a semi-circle.

∴ ABC is a rt. ∠. (III—11.)

∴ ∠ BAC + ∠ BCA = a rt. ∠

= ∠ CAD. (III—16.)

Take away the common ∠ BAC,

∴ ∠ BAD = ∠ ACB,

= ∠ in the segment ACB.

∵ AHBC is an inscribed quadrilateral,

∴ ∠ H + ∠ C = a st. ∠ (III—13.)

= st. ∠ DAE.

But $\angle$ C = $\angle$ BAD.

Take away these equal $\angle$s.

$\therefore$ $\angle$ BAE = $\angle$ H

= $\angle$ in the segment AHB.

120.—Exercises

1. AB is a chord of a circle and AC is a diameter. AD is $\perp$ to the tangent at B. Show that AB bisects the $\angle$ DAC.

2. Two circles intersect at A and B. Any point P on the circumference of one circle is joined to A and B and the joining lines are produced to meet the circumference of the other circle at C, D. Show that CD is || to the tangent at P.

3. LMN is a $\triangle$. Show how to draw the tangent at L to the circumscribed circle without finding the centre of this circle.

4. If either of the $\angle$s which a st. line, drawn through one end of a chord of a circle, makes with the chord is equal to the $\angle$ in the segment on the other side of the chord, the st. line is a tangent. (*Converse of III—18.*)

5. The tangent at a point P on a circle meets the chord MN produced through N, at Q. Prove $\angle$ Q = $\angle$ PNM — $\angle$ PMN.

6. A tangent drawn || to a chord of a circle bisects the arc cut off by the chord.

7. FGE, HKE are two circles, and FEH, GEK two st. lines. Prove that FG, KH meet at an $\angle$ which = the $\angle$ between the tangents to the circles at E.

8. G is the middle point of an arc EGF of a circle. Show that G is equidistant from the chord EF and the tangent at E.

9. A st. line EF is trisected in G, H, and an equilateral $\triangle$ PGH is described on GH. Show that the circle FGP touches EP.

10. D, E, F are respectively the points of contact of the sides MN, NL, LM of a $\triangle$ circumscribed about a circle. DG, EH are respectively $\perp$ EF, DF. Prove GH || LM.

11. The tangent at L to the circumscribed circle of △ LMN meets MN produced at D, and the internal and external bisectors of the ∠ MLN meet MN at E, F respectively. Prove that D is the middle point of EF.

12. GEF, HEF are two circles and GEH is a st. line. The tangents at G, H meet at K. Show that K, G, F, H are concyclic.

13. Points P, Q are taken on two st. lines LM, LN so that LP + LQ = a given st. line. Prove that the circle PLQ passes through a second fixed point.

14. E, F, G, H are the points of contact of the sides XY, YZ, ZV, VX of a quadrilateral circumscribed about a circle. If X, Y, Z, V are concyclic, show that EG ⊥ FH.

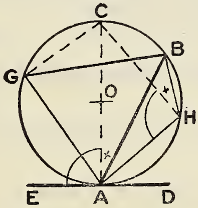
15. XYZV is a quadrilateral inscribed in a circle, and XZ, YV cut at E. Prove that the tangent at E to the circle XEY is || ZV.

16. F is the point of contact of a tangent EF to the circle FGH. GK drawn || EF meets FH, or FH produced, at K. Show that the circle through G, K, H touches FG at G.

17. If from an external point P a tangent PT and a secant PMN are drawn to a circle, the △s PTM, PNT are similar.

18. Use III—18 to prove that the tangents drawn to a circle from an external point are equal.

19. From an external point T a tangent TR and a secant TQP through the centre are drawn to a circle. Prove that ∠ T + 2 ∠ TRQ = a rt. ∠.

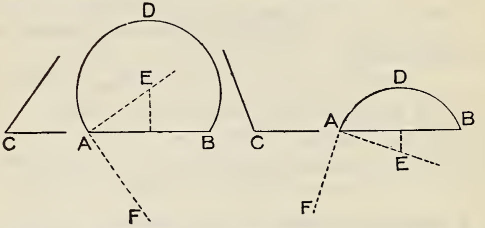


20. Prove Theorem III—18, by using the construction indicated in the diagram and without using III—13.

Constructions

Proposition 19 Problem

On a given straight line construct a segment containing an angle equal to a given angle.



Let AB be the given st. line, and C the given ∠.

Construction.—Make ∠ BAF = ∠ C.

Draw AE ⊥ AF.

Draw the right bisector of AB and produce it to cut AE at E.

∵ E is in the right bisector of AB, it is equidistant from A and B. (Loci. Ex. I.)

With centre E and radius EA describe the arc ADB.

ADB is the required segment.

Proof.—∵ AF is ⊥ AE,

∴ AF is a tangent to the circle ADB.

(III—16, Cor. 1.)

∵ AB is a chord drawn from the point of contact of the tangent AF,

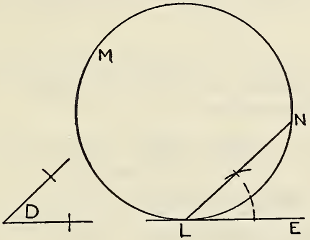
∴ ∠ in segment ADB = ∠ FAB. (III—18.)

But, ∠ FAB = ∠ C, (*Const.*)

∴ ∠ in segment ADB = ∠ C.

Proposition 20 Problem

From a given circle cut off a segment containing an angle equal to a given angle.



Let LMN be the given circle, and D the given ∠.

Construction.—Draw a tangent LE to the given circle.

At L make the ∠ ELN = ∠ D.

LMN is the required segment.

Proof.—∵ LE is a tangent, and LN a chord,

∴ ∠ in segment LMN = ∠ NLE. (III—18.)

But, ∠ NLE = ∠ D. (*Const.*)

∴ ∠ in segment LMN = ∠ D.

121.—Exercises

1. On st. lines each 4 cm. in length, describe segments containing $\angle$s of (*a*) 45°, (*b*) 150°, (*c*) 72°, (*d*) 116°. (*Use the protractor for* (*c*) *and* (*d*).)

2. On a given base construct an isosceles $\triangle$ with a given vertical $\angle$.

3. Divide a circle into two segments such that the $\angle$ in one segment is (*a*) twice, (*b*) three times, (*c*) five times, (*d*) seven times the $\angle$ in the other segment.

4. Construct two $\triangle$s ABC_1, ABC_2 on the same base. $AB = 4$ cm., having $\angle AC_1B = \angle AC_2B = 50°$, and $AC_1 = AC_2 = 5$ cm.

Prove that $\angle ABC_1 + \angle ABC_2 = 2$ rt. $\angle$s.

5. Construct a $\triangle$ LMN having $LM = 5$ cm., $\angle N = 110°$, and the median from $N = 2$ cm.

Measure the greatest and least values the median from N could have, with $LM = 5$ cm., and $\angle N = 110°$.

6. Construct a $\triangle$ having its base 5 cm., its vertical $\angle$ 70°, and its altitude 3 cm.

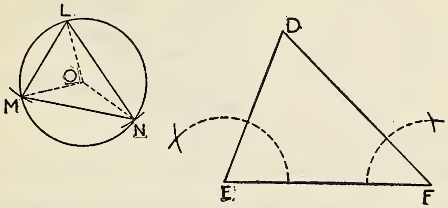
7. Construct a $\triangle$ XYZ, having $XY = 4$ cm., $\angle Z = 40°$, and $XZ + ZY = 10$ cm.

8. Construct a $\triangle$ XYZ, having $XY = 6$ cm., $\angle Z = 50°$ and $XZ - ZY = 4$ cm.

9. Through a given point draw a st. line to cut off from a given circle a segment containing an $\angle$ equal to a given $\angle$.

PROPOSITION 21 PROBLEM

In a given circle inscribe a triangle similar to a given triangle.



Let LMN be the given circle, and DEF the given $\triangle$.

Construction.—Draw a radius OL of the circle.

Make $\angle$ LON = 2 $\angle$ E, and $\angle$ LOM = 2 $\angle$ F.

Join LM, MN, NL.

LMN is the required $\triangle$.

Join OM, ON.

Proof.—$\because$ $\angle$ LON at the centre and $\angle$ LMN at the circumference stand on the same arc.

$\therefore$ $\angle$ LON = 2 $\angle$ LMN, (III—8.)

But $\angle$ LON = 2 $\angle$ E, (*Const.*)

$\therefore$ $\angle$ LMN = $\angle$ E.

Similarly $\angle$ LNM = $\angle$ F.

$\because$ $\angle$ LMN = $\angle$ E,

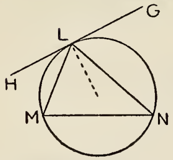
and $\angle$ LNM = $\angle$ F,

$\therefore$ $\angle$ MLN = $\angle$ D.

and $\therefore$ $\triangle$ LMN ||| $\triangle$ DEF.

122.—Exercises

1. Prove the following construction for inscribing a $\triangle$ similar to a given $\triangle$ **DEF** in the circle **LMN**.



Draw a tangent **HLG**. Make $\angle$ **GLN** = $\angle$ **E**, and $\angle$ **HLM** = $\angle$ **F**. Join **MN**.

2. Inscribe an equilateral $\triangle$ in a given circle.

3. Inscribe a square in a given circle.

4. Inscribe a regular pentagon in a given circle. (Use protractor).

5. Inscribe a regular hexagon in a given circle. (Without protractor).

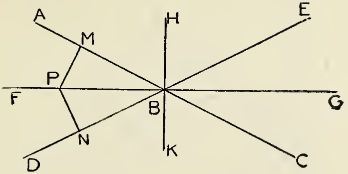
6. Inscribe a regular octagon in a given circle.

7. Two $\triangle$s **LMN, DEF,** each similar to a given $\triangle$ **GHK,** are inscribed in a given circle. Prove $\triangle$ **LMN** $\equiv$ $\triangle$ **DEF.**

8. In a given circle inscribe a $\triangle$ having its sides || to the sides of a given $\triangle$.

Proposition 22 Problem

Find the locus of the centres of circles touching two given intersecting straight lines.



Let ABC, DBE be the two st. lines.

Construction.—Draw the bisectors FBG, HBK of the ∠s made by AC and DE.

These bisectors make up the required locus.

Proof.—Take a point P in either FG or HK, and draw PM ⊥ AC, PN ⊥ DE.

In △s PMB, PNB, $\begin{cases} \angle \text{PBM} = \angle \text{PBN}, \\ \angle \text{PMB} = \angle \text{PNB}, \\ \text{and PB is common,} \end{cases}$

∴ PM = PN. (I—4.)

Hence, a circle described with centre P and radius PM will pass through N.

∵ ∠s at M, N are rt. ∠s,

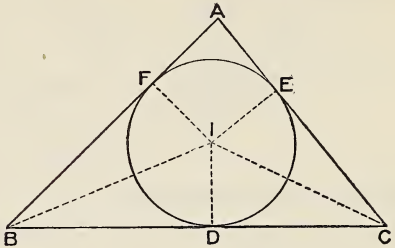
∴ AC, DE are tangents to the circle. (III—16, Cor. 1.)

123. **Definitions.**—When a circle is within a triangle, and the three sides of the triangle are tangents to the circle, the circle is said to be **inscribed in the triangle,** and is called the **inscribed circle of the triangle.**

When a circle lies without a triangle, and touches one side and the other two sides produced, the circle is called an **escribed circle of the triangle.**

Proposition 23 Problem

Inscribe a circle in a given triangle.



Let ABC be the given △.

Bisect ∠s B and C and produce the bisectors to meet at I.

Draw ID, IE, IF, ⊥ BC, CA, AB respectively.

In △s BID, BIF, $\begin{cases} \angle \text{IBD} = \angle \text{IBF}, \\ \angle \text{IDB} = \angle \text{IFB}, \\ \text{IB is common,} \end{cases}$

∴ ID = IF. (I—4.)

Similarly, ID = IE.

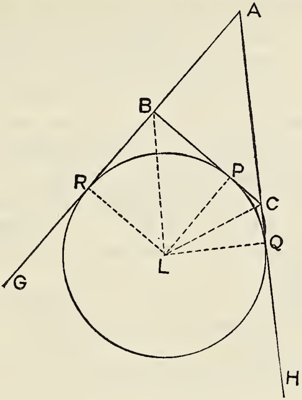
∴ a circle described with centre I and radius ID will pass through E and F.

And ∵ the ∠s at D, E and F are rt. ∠s,

∴ the circle will touch BC, CA and AB. (III—16, Cor. 1.)

Proposition 24 Problem

Draw an escribed circle of a given triangle.



Let ABC be a given $\triangle$ having AB, AC produced to G, H.

It is required to describe a circle touching the side BC and the two sides AB, AC produced.

Bisect $\angle$s GBC, HCB and let the bisectors meet at L. Draw $\perp$s LP, LQ, LR to BC, CH, BG respectively.

In $\triangle$s LBP, LBR, $\begin{cases} \angle \text{PBL} = \angle \text{RBL}, \\ \angle \text{LPB} = \angle \text{LRB}, \\ \text{LB is common,} \end{cases}$

$\therefore$ LP = LR. (I—4.)

Similarly LP = LQ.

$\therefore$ a circle described with centre L and radius LP will pass through R and Q.

$\because$ the $\angle$s at P, Q and R are rt. $\angle$s,

$\therefore$ the circle will touch BC, and CA and AB produced. (III—16, Cor. 1.)

124.—Exercises

1. Make an $\angle$ YXZ $= 45°$. Find a point P such that its distance from XY is 3 cm., and its distance from XZ is 4 cm.

2. Make an $\angle$ YXZ $= 60°$. Find a point P such that its distance from XY is 4 cm., and its distance from XZ is 5 cm.

3. The bisectors of the $\angle$s of a $\triangle$ are concurrent.

4. The bisectors of the exterior $\angle$s at two vertices of a $\triangle$ and the bisector of the interior $\angle$ at the third vertex are concurrent.

5. If a, b, c represent the numerical measures of the sides BC, CA, AB respectively of $\triangle$ ABC, and $s = \frac{1}{2}(a + b + c)$,

(a) AF $= s - a$, BD $= s - b$, CE $= s - c$, when D, E and F are the points of contact of BC, CA, AB with the inscribed circle. (Diagram of III—23.)

(b) AR $= s$, BP $= s - c$, CP $= s - b$, where R and P are the points of contact of AB produced and of BC with an escribed circle. (Diagram of III—24.)

(c) If r be the radius of the inscribed circle, rs = the area of $\triangle$ ABC.

(d) If r_1 be the radius of the escribed circle touching BC, $r_1(s - a)$ = the area of $\triangle$ ABC.

6. If the base and vertical $\angle$ of a $\triangle$ be given, find the locus of the inscribed centre.

7. If the base and vertical $\angle$ of a $\triangle$ are given, find the loci of the escribed centres.

8. L, M, N are the centres of the escribed circles of $\triangle$ PQR. Show that the sides of $\triangle$ LMN pass through the vertices of $\triangle$ PQR.

9. If the centres of the escribed circles are joined, and the points of contact of the inscribed circle with the sides are joined, the $\triangle$s thus formed are similar.

10. Construct a $\triangle$ having given the base, the vertical $\angle$ and the radius of the inscribed circle.

11. Describe a circle cutting off three equal chords of given length from the sides of a given $\triangle$.

12. An escribed circle of $\triangle$ **ABC** touches **BC** at **D** and also touches **AB** and **AC** produced. The inscribed circle touches **BC** at **E**. Show that **DE** equals the difference of **AB** and **AC**.

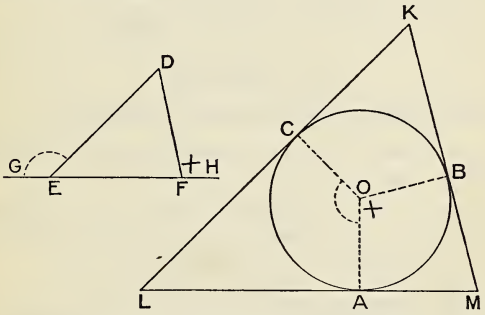
13. Circumscribe a square about a given circle.

14. Inscribe a circle in a given square.

15. Circumscribe a circle about a given square.

Proposition 25 Problem

About a given circle circumscribe a triangle similar to a given triangle.



Let **ABC** be the given circle and **DEF** the given $\triangle$.

Construction.—Produce **EF** to **G** and **H**.

Draw any radius **OA** of the circle, and at **O** make $\angle$ **AOB** = $\angle$ **DFH**, and $\angle$ **AOC** = $\angle$ **DEG**; and produce the arms to cut the circle at **B**, **C**.

At **A**, **B**, **C** draw tangents to the circle meeting at **K**, **L** and **M**.

KLM is the required $\triangle$.

Proof.—$\because$ $\angle$s MAO and MBO in the quadrilateral MBOA are rt. $\angle$s,

$$\therefore \angle M + \angle AOB = 2 \text{ rt. } \angle\text{s}.$$
$$= \angle DFE + \angle DFH.$$

But, $\angle AOB = \angle DFH$,

$$\therefore \angle M = \angle DFE.$$

Similarly, $\angle L = \angle DEF$.

$$\therefore \angle L + \angle M = \angle DEF + \angle DFE,$$

and $\therefore \angle K = \angle EDF$.

$$\therefore \triangle KLM \parallel\!| \triangle DEF.$$

125.—Exercises

1. About a given circle circumscribe an equilateral $\triangle$.

2. If two similar $\triangle$s are circumscribed about the same circle, the $\triangle$s are congruent.

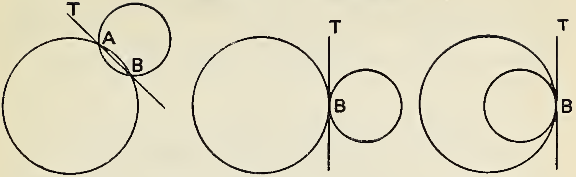
3. Describe a $\triangle$ LMN similar to a given $\triangle$ and such that a given circle is touched by MN and by LM and LN produced.

4. Describe a circle to touch three given straight lines.

5. About a given circle describe a quadrilateral equiangular to a given quadrilateral.

6. If an equilateral triangle be described about a circle and the points of contact be joined, the triangle so formed is also equilateral.

126 CONTACT OF CIRCLES



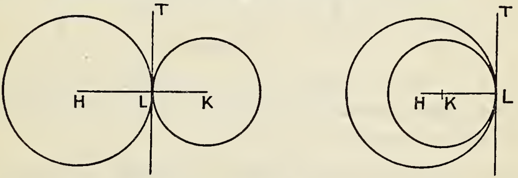
Suppose two circles intersect (as in Fig. 1) in the points A and B. Keeping one circle fixed in position let the other turn about the point B in such a way that A continually approaches B. Ultimately, when A and B coincide the circles are said to touch each other at B (as in Fig. 2 and Fig. 3).

Evidently the secant TAB ultimately passes through two coincident points on each circle and therefore is a tangent to each circle.

Hence, **if two circles touch they have a common tangent at the point of contact.**

PROPOSITION 26 THEOREM

If two circles touch each other, the straight line joining their centres passes through the point of contact.



Hypothesis.—Two circles with centres H and K meet at L.

Prove that the st. line HK passes through L.

Proof.—Since the given circles touch at L, they have a common tangent at that point. (Art. 126.)

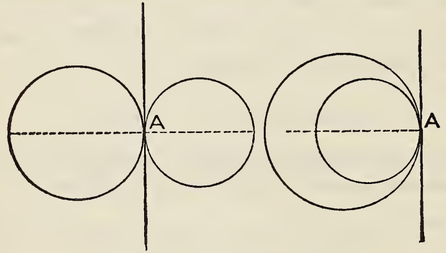
Since HL and KL are radii drawn to the point of contact,

$\because$ HL and KL are both $\perp$ TL, (III—16.)

$\therefore$ HL and KL are in the same straight line.

Hence the line joining the centres of two circles which touch passes through the point of contact.

Definition.—If two circles which touch each other are on opposite sides of the common tangent at their point of contact, and consequently each circle outside the other, they are said to touch **externally**; if they are on the same side of the common tangent, and consequently one within the other, they are said to touch **internally.**



Cor. 1.—If two circles touch externally, the distance between their centres is equal to the sum of their radii; and conversely.

Cor. 2.—If two circles touch internally, the distance between their centres is equal to the difference of their radii; and conversely.

127. —Exercises

1. If the st. line joining the centres of two circles pass through a point common to the two circumferences, the circles touch each other at that point.

2. Find the locus of the centres of all circles which touch a given circle at a given point.

3. Draw three circles with radii 23, 32 and 43 mm. each of which touches the other two externally.

4. Draw a circle of radius 9 cm., and within it draw two circles of radii 3 cm. and 4 cm., to touch each other externally, and each of which touches the first circle internally.

5. Draw a circle of radius 85 mm., and within it draw two circles of radii 25 mm. and 35 mm., to touch each other externally, and each of which touches the first circle internally.

6. Draw a △ **ABC** with sides 5, 12 and 13 cm. Draw three circles, with centres **A**, **B** and **C** respectively, each of which touches the other two externally.

7. Construct the △ **ABC**, having $a = 5$ cm., $b = 4$ cm., and $c = 3$ cm. Draw three circles with centres **A**, **B** and **C** respectively, such that the circles with centres **B** and **C** touch externally, and each touches the circle with centre **A** internally.

8. Mark two points **P** and **Q** 10 cm. apart. With centres **P** and **Q**, and radii 4 cm. and 3 cm., describe two circles. Draw a circle of radius 5 cm. which touches each of the first two circles externally. Find the distance of the centre from **PQ**.

9. Describe a circle to pass through a given point and touch a given circle at a given point.

10. If two circles touch each other, any st. line drawn through the point of contact will cut off segments that contain equal ∠s.

11. Two circles **ACO, BDO** touch, and through **O**, st. lines **AOB, COD** are drawn. Show that **AC** || **BD.**

12. If two || diameters be drawn in two circles which touch each other, the point of contact and an extremity of each diameter are in the same st. line.

13. Describe a circle which shall touch a given circle, have its centre in a given st. line, and pass through a given point in the st. line.

14. Describe three circles having their centres at three given points and touching one another in pairs. Show that there are four solutions.

15. Two circles touch a given st. line at two given points, and also touch each other; find the locus of their point of contact.

16. If through the point of contact of two touching circles a st. line be drawn cutting the circles again at two points, the radii drawn to these points are ||.

17. In a given semi-circle inscribe a circle having its radius equal to a given st. line.

18. Inscribe a circle in a given sector.

19. A circle of 2·5 cm. radius has its centre at a distance of 5 cm. from a given st. line. Describe four circles each of 4 cm. radius to touch both the circle and the st. line.

20. If **DE** be drawn || to the base **GH** of a △ **FGH** to meet **FG, FH** at **D, E** respectively, the circles described about the △s **FGH, FDE** touch each other at **F.**

21. Two circles with centres **P, Q** touch externally and a third circle is drawn, with centre **R,** which both the first circles touch internally. Prove that the perimeter of △ **PQR** = the diameter of the circle with centre **R.**

Miscellaneous Exercises

1. If two chords of a circle intersect at rt. ∠s, the sum of the squares on their segments is equal to the square on the diameter.

2. Find a point in the circumference of a given circle the sum of the squares on whose distances from two given points may be a maximum or minimum.

3. **AOB, COD** are chords cutting at a point **O** within the circle. Show that ∠ **BOC** equals an ∠ at the circumference, subtended by an arc which is equal to the sum of the arcs subtending ∠s **BOC, AOD.**

4. Two chords **AB, CD** intersect at a point **O** without a circle. Show that ∠ **AOC** equals an ∠ at the circumference subtended by an arc which is equal to the difference of the two arcs **BD, AC** intercepted between **OBA** and **ODC.**

5. Two circles touch externally at **E,** and are cut by a st. line at **A, B, C, D.** Show that ∠ **AED** is supplementary to ∠ **BEC.**

6. If at a point of intersection of two circles the tangents drawn to the circles are at rt. ∠s, the st. line joining the points where these tangents meet the circles again, passes through the other point of intersection of the circles.

7. Find a point within a given △ at which the three sides subtend equal ∠s. When is the solution possible?

8. Through one of the points of intersection of two given circles draw the greatest possible st. line terminated in the two circumferences.

9. Through one of the points of intersection of two given circles draw a st. line terminated in the two circumferences and equal to a given st. line.

10. Describe a circle of given radius to touch two given circles.

11. DEF is a st. line cutting BC, CA, AB, the sides of △ ABC, at D, E, F respectively. Show that the circles circumscribed about the △s AEF, BFD, CDE, ABC, all pass through one point.

12. Two circles touch each other at A and BAC is drawn terminated in the circumferences at B, C. Show that the tangents at B, C are ||.

13. D, E, F are any points on the sides BC, CA, AB of △ ABC. Show that the circles circumscribed about the △s AFE, BDF, CED pass through a common point.

14. Two arcs stand on a common chord AB. P is any point on one arc and PA, PB cut the other arc at C, D. Show that the length of CD is constant.

15. ACB is an ∠ in a segment. The tangent at A is || to the bisector of ∠ ACB and meets BC produced at D. Show that AD = AB.

16. Describe a circle of given radius to touch two given intersecting st. lines.

17. In the △ ABC, the bisector of ∠ A meets BC at D. O is the centre of a circle which touches AB at A and passes through D. Prove that OD ⊥ AC.

18. The st. line BC of given length moves so that B and C are respectively on two given fixed st. lines AX and AY. Prove that the circumcentre of △ ABC lies on the circumference of a circle with centre A.

19. ABC is an isosceles △ in which AB = AC. D is any point in BC. Show that the centre of the circle ABD is the same distance from AB that the centre of the circle ACD is from AC.

20. **E, F, G, H** are the points of contact of the sides of a quadrilateral **ABCD** circumscribed about a circle. Prove that the difference of two opposite $\angle$s of **ABCD** = twice the difference of two adjacent $\angle$s of **EFGH.**

21. **ABC** is a $\triangle$ in which **AX, BY** are $\perp$ **BC, CA** respectively. Prove that the tangent at **X** to the circle **CXY** passes through the middle point of **AB**; and the tangent at **C** to the same circle $\parallel$ **AB.**

22. The inscribed circle of $\triangle$ **ABC** touches **BC** at **D.** Prove that the circles inscribed in $\triangle$s **BAD, CAD** touch each other.

23. **O** is the circumcentre of the $\triangle$ **ABC**, and **AO, BO, CO** produced meet the circumference in **D, E, F.** Prove $\triangle$ **DEF** $\equiv \triangle$ **ABC.**

24. **ABC** is a rt.-$\angle$d $\triangle$, **A** being the rt. $\angle$. Prove that **BC** = the difference between the radius of the inscribed circle and the radius of the circle which touches **BC** and the other two sides produced.

25. Describe two circles to touch two given circles, the point of contact with one of these given circles being given.

26. Circles through two fixed points **A** and **B** intersect fixed st. lines, which terminate at **A** and are equally inclined to **AB** on opposite sides of it, in the points **L, M.** Prove that **AL** + **AM** is constant.

27. **AB** is a diameter and **CD** a chord of a given circle. **AX** and **BY** are both $\perp$ **CD.** Prove that **CX** = **DY.**

28. Through a fixed point **A** on a circle any chord **AB** is drawn and produced to **C** making **BC** = **AB.** Find the locus of **C.**

29. Construct a $\triangle$ having given the base, the vertical $\angle$, and the length of the median drawn from one end of the base.

30. If the sum of one pair of opposite sides of a quadrilateral be equal to the sum of the other pair, a circle may be inscribed in the quadrilateral.

31. Construct a $\triangle$ having given the vertical $\angle$, the base, and the point where the bisector of the vertical $\angle$ cuts the base.

32. From the ends of a diameter **BC** of a circle, $\parallel$ chords **BE, CF** are drawn, meeting the circle again in **E** and **F.** Prove that **EF** is a diameter.

33. **ACFB** and **ADEB** are fixed circles; **CAD, CBE** and **DBF** are st. lines. Prove that **CF** and **DE** meet at a constant $\angle$.

34. **A, B, C, D** are four points in order on the circumference of a circle, and the arc **AB** = the arc **CD.** If **AC** and **BD** cut at **E,** the chord which bisects $\angle$s **AEB, CED** is itself bisected at **E.**

35. **AB, AC** are tangents at **B, C** to a circle, and **D** is the middle point of the minor arc **BC.** Prove that **D** is the centre of the inscribed circle of the $\triangle$ **ABC.**

36. Construct an equilateral $\triangle$ whose side is of given length so that its vertices may be on the sides of a given equilateral $\triangle$.

37. **D, E, F** are the points of contact of the sides **BC, CA, AB** of a $\triangle$ **ABC** with its inscribed circle. **FK** is $\perp$ **DE,** and **EH** is $\perp$ **FD.** Prove **HK** $\parallel$ **BC.**

38. Tangents are drawn from a given point to a system of concentric circles. Find the locus of their points of contact.

39. From a given point A without a given circle draw a secant ABC such that AB = BC.

40. EF is a fixed chord of a given circle, P any point on its circumference. EM $\perp$ FP and FN $\perp$ EP. Find the locus of the middle point of MN.

41. K is the middle point of a chord PQ in a circle of which O is the centre. LKM is a chord. Tangents at L, M meet PQ produced at G, H respectively. Prove $\triangle$ OGL $\equiv$ $\triangle$ OHM.

42. LM is the diameter of the semi-circle LNM in which arc LN $>$ arc NM, and ND $\perp$ LM. A circle inscribed in the figure bounded by ND, DM and the arc NM touches DM at E. Show that LE = LN; and hence give a construction for inscribing the circle.

43. GK is a diameter and O the centre of a circle. A tangent KD = KO. From O a $\perp$ OE is drawn to GD. KE is joined and produced to meet the circumference in F. Prove that FE = FG.

44. LPM and LQRM are two given segments on the same chord LM. If P moves on the arc LPM such that LQP and MRP are st. lines, the length of QR is constant.

45. EFP, EFRS are two circles and PFR, PES are st. lines. O is the centre of the circle EFP. Prove that PO $\perp$ RS.

46. E, F are fixed points on the circles EPD, FQD, and PDQ is a variable st. line. PE, QF intersect at R. Find the locus of R.

47. The circle PEGF passes through the centre G of the circle QEF, and P, E, Q are in a st. line. Prove that PQ = PF.

48. Through two points on a diameter equally distant from the centre of a circle, $\parallel$ chords are drawn, show that

these chords are the opposite sides of a rectangle inscribed in the circle.

49. If through the points of intersection of two circles any two || st. lines are drawn and the ends joined towards the same parts, the figure so formed is a ||gm.

50. Any two || tangents are drawn, one to each of two given circles; a st. line is drawn through the points of contact; show that the tangents to the circles at the other points of intersection are also ||.

51. The hypotenuse of a rt.-∠d △ is fixed and the other two sides are moveable; find the locus of the point of intersection of the bisectors of the acute ∠s of the △.

52. From the middle point **L** of the arc **MLN** of a circle two chords are drawn cutting the chord **MN** and the circumference. Show that the four points of intersection are concyclic.

53. If from one end of a diameter of a circle two st. lines are drawn to the tangent at the other end of the diameter, the four points of intersection—with the circle, and with the tangent—are concyclic.

54. **ABC** is a diameter of a circle, **B** being the centre. **AD** is a chord, and **BE** ⊥ to **AC** cutting the chord at **E.** Show that **BCDE** is a cyclic quadrilateral; and that the circles described about **ABE** and the quadrilateral **BCDE,** are equal.

55. Two circles intersect at **A** and **B.** From **A** two chords **AC** and **AE** are drawn one in each circle making equal ∠s with **AB,** st. lines **CBD** and **EBF** are drawn to cut the circles at **D** and **F,** prove **C, F, D, E** concyclic; also prove △s **FCA** and **DEA** similar.

56. **ABC** is a △ and any circle is drawn passing through **B,** and cutting **BC** at **D** and **AB** at **F**; another circle is

drawn passing through **C** and **D** and intersecting the former circle at **E** and **AC** at **G.** Prove **A, F, E, G** are concyclic.

57. If two equal circles intersect, the four tangents at the points of intersection form a rhombus.

58. If two equal circles cut, and at **G**, one of the points of intersection, chords be drawn in each circle to touch the other circle, these chords are equal.

59. Two equal circles, centres **O** and **P**, touch externally at **S**, **SQ** and **SR** are drawn $\perp$ to each other cutting the circumferences at **Q** and **R** respectively. Show that **O, P, Q** and **R** are the vertices of a ||gm.

60. **AB, CD,** and **EF** are || chords in a circle, prove that the $\triangle$s **ACE** and **BDF** are congruent; also **ACF** and **BDE**; also **ADF** and **BCE.**

61. On the circumference of a circle are two fixed points which are joined to a moveable point either inside or outside the circle. If these lines intercept a constant arc, find the locus of the point.

62. **KL** is any chord of a circle and **H** the middle point of one of the arcs, any st. line **HED** cuts **KL** at **E** and the circumference at **D.** Show that **HL** is a tangent to the circle about **LED,** and **HK** a tangent to that about **KED.**

63. Two circles intersect at **E** and **F.** From any point **P** on the circumference of one of them st. lines **PE** and **PF** are drawn to meet the circumference of the other at **Q** and **R**; show that the length of the straight line **QR** is constant. [Take **P** both on the major arc and on the minor arc.]

64. **HKL** is a $\triangle$ having $\angle$ **H** acute; on **KL** as diameter a circle, centre **O,** is described cutting **HK** at **D** and **HL** at **E.** Show that $\angle$ **ODE** = $\angle$ **H.**

65. P is a point external to two concentric circles whose centre is O, PQ is a tangent to the outer circle and PR and PS are tangents to the inner circle. Show that ∠ RQS is bisected by QO.

66. If the extremities of two ∥ diameters in two circles are joined by a st. line which cuts the circles, the tangents at the points of intersection are ∥. Show that this is true for the four cases that arise.

67. KLMN is a ∥gm, through L and N two ∥ st. lines are drawn cutting MN at F and KL at E, show that the circles described about the △s KNE and LMF are equal.

68. EFGH is a quadrilateral having EF ∥ HG and EH = FG. From E a st. line EK is drawn ∥ FG meeting HG at K. Show that circles described about the △s EHG, EKG are equal.

69. From any point P on the circumference of a circle PD, PE and PF are perpendiculars to a chord QR, and to the tangents QT and RT. Show that the △s PED and PFD are similar.

70. A quadrilateral having two ∥ sides is described about a circle. Show that the st. line drawn through the centre ∥ to the ∥ sides and terminated by the nonparallel sides is one quarter of the perimeter of the quadrilateral.

71. CD is a diameter of a circle centre O; chords CF and DG intersect within the circle at E. Show that OF is a tangent to the circle passing through F, G and E.

72. EF is a chord of a circle and EP a tangent; a st. line PG ∥ to EF meets the circle at G; prove that the △s EFG and EPG are similar.

73. The diagonals of a quadrilateral are ⊥; show that the st. lines joining the feet of the perpendiculars from

the intersection of the diagonals on the sides form a cyclic quadrilateral.

74. Two chords of a circle intersect at rt. ∠s and tangents are drawn to the circle from the extremities of the chords; show that the resulting quadrilateral is cyclic.

75. A quadrilateral is described about a circle and its vertices are joined to the centre cutting the circumference in four points. Show that the diagonals of the quadrilateral formed by joining these four points are ⊥.

76. **DEF** is a △ inscribed in a circle whose centre is **O**. On **EF** any arc of a circle is described and **ED**, **FD**, or these lines produced, meet the arc at **P**, **Q**. Show that **OD**, or **OD** produced, cuts **PQ** at rt. ∠s.

77. **PQRS** is a ||gm and the diagonals intersect at **E**. Show that the circles described about **PES** and **QER** touch each other; and likewise those about **PEQ** and **RES**.

78. Two equal circles intersect at **E** and **F**; with centre **E** and radius **EF** a circle is described cutting the circles at **G** and **H**. Show that **FG** and **FH** are tangents to the equal circles.

79. If from any point on the circumference of a circle perpendiculars are drawn to two fixed diameters, the line joining their feet is of constant length.

80. From the extremities of the diameter of a circle perpendiculars are drawn to any chord. Show that the centre is equally distant from the feet of the perpendiculars.

81. **EF** and **GH** are || chords in a circle, **F** and **H** being towards the same parts; a point **K** is taken on the circumference such that **GF** bisects ∠ **HGK**. Prove **GK** = **EF**.

82. Two circles intersect at **D** and **E**, and **KEL** and **PEQ** are two chords terminated by the circumferences. Show that the △s **DKP** and **DLQ** are similar.

83. If from two points outside a circle, equally distant from the centre and situated on a diameter produced, tangents are drawn to the circle, the resulting quadrilateral is a rhombus.

84. If the arcs cut off by the sides of a quadrilateral inscribed in a circle are bisected and the opposite points are joined, these two lines shall be ⊥. (*Note.—Use Ex. 3.*)

85. **PQ** is a fixed st. line and **PM, QN** are any two ∥ st. lines, **M** and **N** being towards the same parts. The ∠s **MPQ** and **NQP** are bisected by **PR** and **QR**. Find the locus of **R**.

86. If the ∠s of a △ inscribed in a circle are bisected by lines which meet the circumference, and a new △ be formed by joining these points on the circumference, its sides shall be ⊥ to the bisectors.

87. If two circles touch each other internally, and a st. line be drawn ∥ to the tangent at the point of contact, the two intercepts between the circumferences subtend equal ∠s at the point of contact.

88. **ABC** is a △ inscribed in a circle and **BA** is produced to **E**; **D** is any point in **AE**; circles are described through **B, C, D** and through **B, C, E**; **CFDG** cuts the circles **ABC, EBC** in **F** and **G**. Prove that △s **ADF** and **DEG** are similar.

89. Draw a tangent to a circle which shall bisect a given ∥gm which is outside the circle.

90. In a given circle draw a chord of fixed length which shall be bisected by a given chord.

91. In a given circle draw a chord which shall pass through a given point and be bisected by a given chord. How many such chords can be drawn?

92. Describe a circle with given radius to touch a given st. line and have its centre in another given st. line.

93. Describe a circle of given radius to pass through a given point and touch a given st. line.

94. Describe a circle to touch a given circle at a given point and a given st. line.

95. In a given st. line find a point such that the st. lines joining it to two given points may be (a) $\perp$s, (b) make a given $\angle$ with each other.

96. Describe a circle of given radius to touch a given circle and a given st. line.

97. Describe a circle to touch a given circle and a given st. line at a given point.

BOOK IV.

RATIO AND PROPORTION

128. **Definitions.**—The **ratio** of one magnitude to another *of the same kind* is the number of times that the first contains the second; or it is the part, or fraction, that the first magnitude is of the second.

Thus, the ratio of one magnitude to another is the same as the measure of the first when the second is taken as the unit.

If a st. line be 5 cm. in length, the ratio of its length to the length of one centimetre is 5, that is, the st. line is to one centimetre as 5 is to 1.

If two st. lines A, B are respectively 8 inches and 3 inches in length, then the ratio of A to B is 8 to 3.

The ratio of one magnitude A to another B is written either $\frac{A}{B}$, or $A : B$.

When the form $\frac{A}{B}$ is used, the upper magnitude is called the **numerator,** and the lower the **denominator;** and when the form $A : B$ is used, the first magnitude is called the **antecedent,** and the second the **consequent.** The two magnitudes are called the **terms** of the ratio.

129. **Definitions.**—**Proportion** is the equality of ratios, *i.e.*, when two ratios are equal to each other, the four magnitudes are said to be in proportion.

The equality of the ratios of K to L and of M to N may be written in any one of the three forms:—
$\frac{K}{L} = \frac{M}{N}$, K : L = M : N or K : L : : M : N; and is read "K is to L as M is to N."

The four magnitudes in a proportion are called **proportionals.**

The first and the last are called the **extremes;** the second and the third are called the **means.**

The first two magnitudes of a proportion must be of the same kind, and the last two must be of the same kind; but the first two need not be of the same kind as the last two. Thus in the proportion $\frac{D}{E} = \frac{F}{H}$, D and E may be lengths of lines, while F and H are areas.

130. **Definitions.**—Three magnitudes are said to be **in continued proportion,** or in **geometric progression,** when the ratio of the first to the second equals the ratio of the second to the third.

Three magnitudes L, M, N, of the same kind, are in continued proportion, if $\frac{L}{M} = \frac{M}{N}$.

e.g.:—L = 4 cm., M = 6 cm., N = 9 cm.

The second magnitude of a continued proportion is called the **mean proportional,** or **geometric mean,** of the other two.

131. Two magnitudes of the same kind are **commensurable** when each contains some common measure an integral number of times.

132. Two magnitudes of the same kind are **incommensurable** when there is no common measure, however small, contained in each of them an integral number of times.

The diagonal and the side of a square are incommensurable, the ratio of the diagonal to the side being $\sqrt{2} : 1$.

The side of an equilateral triangle and the perpendicular from a vertex to the opposite side are incommensurable, the ratio of a side to the perpendicular being $2 : \sqrt{3}$.

$\sqrt{2} = 1{\cdot}414$ nearly, and $\sqrt{3} = 1{\cdot}732$ nearly, but while these roots may be calculated to any required degree of accuracy they cannot be exactly found. Thus, there is no straight line however short that is contained an integral number of times in both the diagonal and the side of a square; or in both the side and the altitude of an equilateral triangle.

The treatment of incommensurable magnitudes is too difficult for an elementary text-book, but as in algebra, the relations that are obtained in geometry for commensurable magnitudes hold good also for incommensurable magnitudes.

133. The following simple algebraic theorems are used in geometry:—

1. If $\frac{a}{b} = \frac{c}{d}$, $ad = bc$.

If four numbers be in proportion, the product of the extremes is equal to the product of the means.

2. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a}{c} = \frac{b}{d}$.

If four numbers be in proportion, the first is to the third as the second is to the fourth.

When a proportion is changed in this way, the second proportion is said to be formed from the first by **alternation.**

In order that a given proportion may be changed by alternation, the four magnitudes must be of the same kind.

e.g.:—$\frac{2\text{ ft.}}{5\text{ ft.}} = \frac{4\text{ ft.}}{10\text{ ft.}}$ and, by alternation, $\frac{2\text{ ft.}}{4\text{ ft.}} = \frac{5\text{ ft.}}{10\text{ft.}}$;

but from the proportion $\frac{\text{st. line } \mathsf{D}}{\text{st. line } \mathsf{E}} = \frac{\text{area } \mathsf{F}}{\text{area } \mathsf{G}}$ another proportion cannot be inferred by alternation.

3. If $\frac{a}{b} = \frac{c}{d}$, $\frac{b}{a} = \frac{d}{c}$.

If four numbers are in proportion, the second is to the first as the fourth is to the third.

When a proportion is changed in this way, the second proportion is said to be formed from the first by **inversion.**

4. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{b} = \frac{c+d}{d}$.

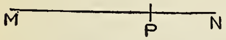
If four numbers are in proportion, the sum of the first and the second is to the second as the sum of the third and the fourth is to the fourth.

5. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a-b}{b} = \frac{c-d}{d}$.

If four numbers are in proportion, the difference of the first and the second is to the second as the difference of the third and the fourth is to the fourth.

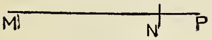
6. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

If four numbers are in proportion, the sum of the first and second terms is to the difference of the first and second terms as the sum of the third and fourth terms is to the difference of the third and fourth terms.



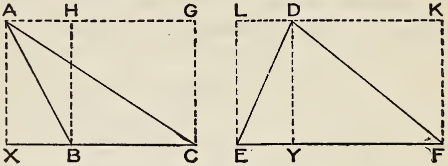
134. If a given straight line MN be divided internally at a point P, the internal segments PM, PN are the distances from P to the ends of the given straight line.

Similarly, if a point P be taken in a given straight line MN produced, the distances from P to the ends of the given straight line, PM, PN, are called the **external segments of the straight line,** or the given straight line is said to be divided **externally** at the point P.



PROPOSITION 1 THEOREM

If triangles have equal altitudes, their areas are to one another as the bases of the triangles.



Hypothesis.—In $\triangle$s ABC, DEF; AX $\perp$ BC, DY $\perp$ EF and AX = DY.

Prove that $\dfrac{\triangle\ ABC}{\triangle\ DEF} = \dfrac{BC}{EF}$.

Construction.—On BC and EF construct the rectangles HC and LF, having HB = AX and LE = DY.

Proof.—Let BC and EF contain a and b units of length respectively, and AX or DY contain c units.

$$\triangle\ ABC = \tfrac{1}{2}\ HB.BC = \tfrac{1}{2}\ ca. \qquad \text{(II—13.)}$$

$$\triangle\ DEF = \tfrac{1}{2}\ LE.EF = \tfrac{1}{2}\ cb.$$

$$\therefore \frac{\triangle\ ABC}{\triangle\ DEF} = \frac{\frac{1}{2}\,ca}{\frac{1}{2}\,cb} = \frac{a}{b} = \frac{BC}{EF}.$$

135.—Exercises

1. $\triangle$s on equal bases are to each other as their altitudes.

2. If two $\triangle$s are to each other as their bases, their altitudes must be equal.

3. ||gms of equal altitudes are to each other as their bases.

4. Construct a $\triangle$ equal to $\frac{5}{4}$ of a given $\triangle$.

5. Construct a ||gm equal to $\frac{5}{2}$ of a given ||gm.

6. **ABC, DEF** are two $\triangle$s having **AB** = **DE** and $\angle$ **B** = $\angle$ **E**. Show that $\triangle$ **ABC** : $\triangle$ **DEF** = **BC** : **EF.**

7. The rectangle contained by two st. lines is a mean proportional between the squares on the lines.

8. If two equal $\triangle$s are on opposite sides of the same base, the st. line joining their vertices is bisected by the common base, or the base produced.

9. The sum of the $\perp$s from any point in the base of an isosceles $\triangle$ to the two equal sides equals the $\perp$ from either end of the base to the opposite side.

10. The difference of the $\perp$s from any point in the base produced of an isosceles $\triangle$ to the equal sides equals the $\perp$ from either end of the base to the opposite side.

11. The sum of the $\perp$s from any point within an equilateral $\triangle$ to the three sides equals the $\perp$ from any vertex to the opposite side.

12. If st. lines **AO, BO, CO** are drawn from the vertices of a $\triangle$ **ABC** to any point **O** and **AO,** produced if necessary, cuts **BC** at **D,**

$$\frac{\triangle \mathbf{AOB}}{\triangle \mathbf{AOC}} = \frac{\mathbf{BD}}{\mathbf{DC}}.$$

13. In any $\triangle$ **ABC, F** is the middle point of **AB, E** is the middle point of **AC,** and **BE, CF** intersect at **O.** Show that **AO** produced bisects **BC;** that is, the medians of a $\triangle$ are concurrent.

14. **ABC** is a $\triangle$ and **O** is any point. **AO, BO, CO,** produced if necessary cut **BC, CA, AB** at **D, E, F** respectively; a_1, a_2, b_1, b_2, c_1, c_2, are respectively the numerical measures of **BD, DC, CE, EA, AF, FB.** Show that $a_1\, b_1\, c_1 = a_2\, b_2\, c_2$. (This is known as Ceva's Theorem.)

15. The four $\triangle$s into which a quadrilateral is divided by its diagonals are proportional.

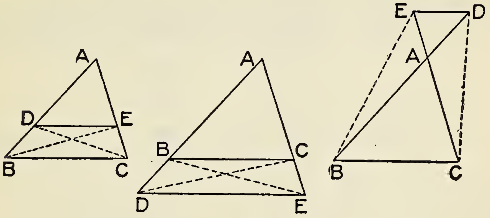
16. **DEF** is a $\triangle$; **G** is a point in **DE** such that **DG** = 3 **GE,** and **H** is a point in **DF** such that **FH** = 3 **HD.** Show that $\triangle$ **FGH** = 9 $\triangle$ **EGH.**

17. St. lines **DG, EH, FK** drawn from the vertices of $\triangle$ **DEF** to meet the opposite sides at **G, H, K** pass through a common point **O.** Prove that $\dfrac{\mathsf{DO}}{\mathsf{DG}} + \dfrac{\mathsf{EO}}{\mathsf{EH}} + \dfrac{\mathsf{FO}}{\mathsf{FK}} = 2.$

18. In $\triangle$ **DEF, G** is taken in side **EF** such that **EG** = 2 **GF,** and **H** is taken in side **FD** such that **FH** = 2 **HD. DG** and **EH** intersect at **O.** Prove that $\dfrac{\triangle\,\mathsf{DOH}}{\triangle\,\mathsf{DEF}} = \dfrac{1}{21}.$

PROPOSITION 2 THEOREM

A straight line drawn parallel to the base of a triangle cuts the sides, or the sides produced, proportionally.



Hypothesis.—In $\triangle$ ABC, DE $\parallel$ BC.

Prove that $\frac{BD}{DA} = \frac{CE}{EA}$.

Construction.—Join BE and DC.

Proof.— $\because$ DE $\parallel$ BC,

$\therefore \triangle BDE = \triangle CDE$ (II—14.)

$$\therefore \frac{\triangle BDE}{\triangle ADE} = \frac{\triangle CDE}{\triangle ADE}.$$

$\because$ $\triangle$s BDE, ADE have the same altitude, viz., the $\perp$ from E to AB,

$$\therefore \frac{\triangle BDE}{\triangle ADE} = \frac{BD}{DA}. \quad \text{(IV—1.)}$$

In the same way,

$$\frac{\triangle CDE}{\triangle ADE} = \frac{CE}{EA}.$$

$$\therefore \frac{BD}{DA} = \frac{CE}{EA}.$$

N.B.—By placing D *on* AB *and* E *on* AC *in all three figures the proof applies to all.*

Cor.—In the first figure,

$$\because \frac{BD}{DA} = \frac{CE}{EA}, \therefore \frac{BD + DA}{DA} = \frac{CE + EA}{EA}. \quad \textit{by addition.}$$

$$\therefore \frac{AB}{AD} = \frac{AC}{AE}.$$

$$\therefore \frac{AD}{AB} = \frac{AE}{AC} \quad \textit{by inverting.}$$

Again,

$$\because \frac{BD}{DA} = \frac{CE}{EA}, \therefore \frac{DA}{BD} = \frac{EA}{CE}. \quad \textit{by inverting.}$$

$$\therefore \frac{DA + BD}{BD} = \frac{EA + CE}{CE}. \quad \textit{by addition.}$$

$$\therefore \frac{AB}{BD} = \frac{AC}{CE}.$$

$$\therefore \frac{BD}{AB} = \frac{CE}{AC}. \quad \textit{by inverting.}$$

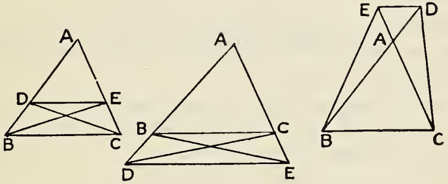
Similar proofs may be given for the second and third figures.

Thus we see that where a line is parallel to the base of a triangle we may form a proportion by taking the whole side or either of the segments, in any order, for the terms of the first ratio, provided we take the corresponding parts of the other side to form the terms of the other ratio in the proportion.

PROPOSITION 3 THEOREM

(Converse of IV—2)

If two sides of a triangle, or two sides produced, are divided proportionally, the straight line joining the points of section is parallel to the base.



Hypothesis.—In $\triangle$ ABC, $\frac{BD}{DA} = \frac{CE}{EA}$.

Prove that DE $\parallel$ BC.

Construction.—Join DC, BE.

Proof.— $\because \quad \frac{BD}{DA} = \frac{CE}{EA}$,

and $\frac{BD}{DA} = \frac{\triangle BDE}{\triangle EDA}$, (IV—1.)

and $\frac{CE}{EA} = \frac{\triangle CDE}{\triangle EDA}$

$\therefore \frac{\triangle BDE}{\triangle EDA} = \frac{\triangle CDE}{\triangle EDA}$.

Hence, $\triangle$ BDE $=$ $\triangle$ CDE

$\therefore$ DE $\parallel$ BC. (II—17.)

136.—Exercises

1. The st. line drawn through the middle point of one side of a △, and || to a second side bisects the third side.

2. The st. line joining the middle points of two sides of a △ is || to the third side.

3. If two sides of a quadrilateral are ||, any st. line drawn || to the || sides and cutting the other sides, will cut these other sides proportionally.

4. **ABCD** is a quadrilateral having **AB** || **DC**. **P, Q** are points in **AD, BC** respectively such that **AP : PD = BQ : QC**. Show that **PQ** || **AB** or **DC**.

5. If two st. lines are cut by a series of || st. lines, the intercepts on one are proportional to the corresponding intercepts on the other.

6. **D, E** are points in **AB, AC**, the sides of △ **ABC**, such that **DE** || **BC**; **BE, CD** meet at **F**. Show that △ **ADF** = △ **AEF**.

Show also that **AF** bisects **DE** and **BC**.

7. Through **D**, any point in the side **BC** of △ **ABC, DE, DF** are drawn || **AB, AC** respectively and meeting **AC, AB** at **E, F**. Show that △ **AEF** is a mean proportional between △s **FBD, EDC**.

8. **ACB, ADB** are two △s on the same base **AB**. **E** is any point in **AB**. **EF** is || **AC** and meets **BC** at **F**. **EG** is || **AD** and meets **BD** at **G**. Prove **FG** || **CD**.

9. **D** is a point in the side **AB** of △ **ABC**; **DE** is drawn || **BC** and meets **AC** at **E**; **EF** is drawn || **AB** and meets **BC** at **F**. Show that **AD : DB = BF : FC**.

10. From a given point **M** in the side **DE** of △ **DEF**, draw a st. line to meet **DF** produced at **N** so that **MN** is bisected by **EF**.

11. **PQRS** is a ||gm, and from the diagonal **PR** equal lengths **PK, RL** are cut off. **SK, SL** when produced meet **PQ, RQ** respectively at **E, F**. Prove **EF** || **PR**.

12. **DEF** is a △ in which **K, M** are points in the side **DE** and **L, N** are points in the side **DF** such that **KL** and **MN** are both || **EF**. Find the locus of the intersection of **KN** and **LM**.

13. **O** any point within a quadrilateral **PQRS** is joined to the four vertices and in **OP** any point **X** is taken. **XY** is drawn || **PQ** to meet **OQ** at **Y**; **YZ** is drawn || **QR** to meet **OR** at **Z**; and **ZV** is drawn || **RS** to meet **OS** at **V**. Prove that **XV** || **PS**.

14. **O** is a fixed point and **P** moves along a fixed st. line. **Q** is a point in **OP**, or in **OP** produced in either direction, such that **OQ** : **QP** is constant. Find the locus of **Q**.

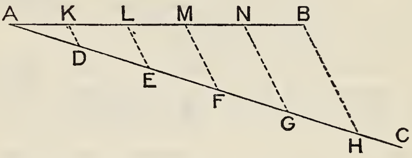
15. **L** is any point in the side **DE** of a △ **DEF**. From **L** a line drawn || **EF** meets **DF** at **M**. From **F** a line drawn || **ME** meets **DE** produced at **N**. Prove that **DL** : **DE** = **DE** : **DN**.

16. If from the vertex of a △ perpendiculars are drawn to the bisectors of the exterior ∠s at the base, the line joining the feet of the perpendiculars is || the base.

Proposition 4 Problem

Divide a given straight line into any number of equal parts.

(*Alternative proof for II—8*)



Let AB be the given straight line.

At A draw AC making any angle with AB and from AC cut off in succession the required number of equal parts. AD, DE, EF, FG, GH.

Join HB and through D, E, F, G draw lines || BH cutting AB at K, L, M, N.

Then AK = KL = LM = MN = NB.

In $\triangle$ AEL, DK || EL,

$$\therefore \frac{AD}{DE} = \frac{AK}{KL}. \qquad \text{(IV—2.)}$$

But AD = DE, $\therefore$ AK = KL.

In $\triangle$ AFM, EL || FM,

$$\therefore \frac{AE}{EF} = \frac{AL}{LM}.$$

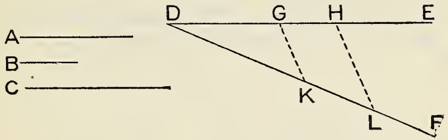
But AE = 2 EF, $\therefore$ AL = 2 LM

$\therefore$ LM = AK or KL.

In the same way it may be proved that AK = KL = LM = MN = NB.

Proposition 5 Problem

Find a fourth proportional to three given straight lines taken in a given order.



Let A, B, C be the three given st. lines.

From a point D draw two st. lines DE, DF.

Cut off DG = A, GH = B, DK = C.

Join GK. Through H draw HL || GK meeting DF in L.

Then KL is the required fourth proportional.

In △ DHL, GK || HL

$$\therefore \frac{DG}{GH} = \frac{DK}{KL} \qquad \text{(IV—2.)}$$

$$\textit{i.e.,} \quad \frac{A}{B} = \frac{C}{KL}$$

∴ KL is the required fourth proportional.

Corollary.—The proposition provides a method for finding a third proportional to two given lines A, B. Repeat the line B and find a fourth proportional to A, B, B. Let this be C.

Then $\frac{A}{B} = \frac{B}{C}$

C is a third proportional to A and B.

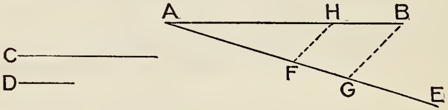
137.—Exercises

1. Find graphically, testing your results by arithmetic,
   1. The fourth proportional to 2″, 1·5″, 1″.
   2. The third proportional to 1·5″, 2·5″.

2. **ABCD** is a parallelogram, **DEF** is drawn cutting **AB** in **E** and **CB** produced in **F**. Show that **CF** is a fourth proportional to **EA**, **AD**, **AB**.

Proposition 6 Problem

Divide a given straight line in a given ratio.



Let **AB** be the given st. line, and $\frac{C}{D}$ the given ratio.

Draw **AE** making any ∠ with **AB**.

On **AE** cut off **AF** = **C**, **FG** = **D**.

Join **BG**, and through **F** draw **FH** || **GB**.

In △ **ABG**, ∵ **FH** || **GB**,

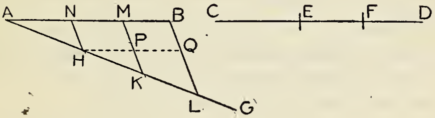
$$\therefore \frac{AH}{HB} = \frac{AF}{FG}. \qquad \text{(IV—2.)}$$

But **AF** = **C**, and **FG** = **D**,

$$\therefore \frac{AH}{HB} = \frac{C}{D}.$$

Proposition 7 Problem

Divide a given straight line similarly to a given divided line.



Let AB be the given st. line, and CD the given line divided at E and F.

At A draw AG making any angle with AB.

From AG cut off AH = CE, HK = EF, KL = FD. Join BL. Through H, K draw HN, KM both || BL.

Then AB is divided at N and M similarly to CD. Through H draw HPQ || AB.

Proof.—In $\triangle$ AMK, NH || MK,

$$\therefore \frac{AN}{NM} = \frac{AH}{HK}. \qquad \text{(IV—2.)}$$

In $\triangle$ HQL, PK || QL,

$$\therefore \frac{HP}{PQ} = \frac{HK}{KL}.$$

But HP = NM and PQ = MB, (II—7.)

$$\therefore \frac{NM}{MB} = \frac{HK}{KL}.$$

$$\therefore \frac{AN}{NM} = \frac{CE}{EF} \text{ and } \frac{NM}{MB} = \frac{EF}{FD}.$$

Both these relations are contained in

$$\frac{AN}{CE} = \frac{NM}{EF} = \frac{MB}{FD}.$$

138.—Exercises

1. Divide the area of a given △ into parts that are in the ratio of two given st. lines.

2. Divide the area of a ||gm into parts that are in the ratio of two given st. lines.

3. Find a third proportional to two given st. lines. Show how two third proportionals, one greater than either of the given st. lines and the other less than either, may be found.

4. Divide a given st. line externally so that the ratio of the segments may equal the ratio of two given st. lines.

5. **BAC** is a given ∠ and **P** is a given point. Through **P** draw a st. line **DPE** cutting **AB** at **D** and **AC** at **E** such that **DP** : **PE** equals the ratio of two given st. lines.

6. Divide a given st. line in the ratio 2 : 3 : 5.

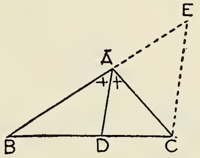
7. Construct a △ having its sides in the ratio 2 : 3 : 4, and its perimeter equal to a given st. line.

8. From a given point **P** outside the ∠ **XOY** draw a line meeting **OX** at **Q** and **OY** at **R** so that **PQ** : **QR** = a given ratio.

Bisector Theorems

Proposition 8 Theorem

If the vertical angle of a triangle be bisected by a straight line which cuts the base, the segments of the base are proportional to the other sides of the triangle.



Hypothesis.—In $\triangle$ ABC, AD bisects $\angle$ BAC.

Prove that $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Construction.—Through C draw CE $\parallel$ AD to meet BA produced at E.

Proof.—AD $\parallel$ EC, $\therefore$ $\angle$ BAD = $\angle$ AEC,

and $\angle$ DAC = $\angle$ ACE.

But $\angle$ BAD = $\angle$ DAC, by hypothesis,

$\therefore$ $\angle$ AEC = $\angle$ ACE.

$\therefore$ AC = AE. (I—12.)

In $\triangle$ EBC, AD $\parallel$ EC,

$\therefore \dfrac{BD}{DC} = \dfrac{BA}{AE}$. (IV—2.)

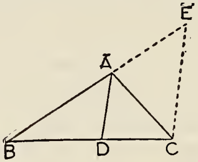
But AE = AC,

$\therefore \dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Proposition 9 Theorem

(*Converse of Proposition 8*)

If the base of a triangle be divided internally into segments that are proportional to the other sides of the triangle, the straight line which joins the point of section to the vertex bisects the vertical angle.



Hypothesis.—In $\triangle$ ABC, $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Prove that AD bisects $\angle$ BAC.

Construction.—Through C draw CE $\parallel$ AD to meet BA produced at E.

Proof.— $\because \dfrac{BD}{DC} = \dfrac{BA}{AC}$, by hypothesis

and $\dfrac{BD}{DC} = \dfrac{BA}{AE}$. (IV—2.)

$\therefore$ AE $=$ AC,

and $\angle$ AEC $= \angle$ ACE.

But $\angle$ AEC $= \angle$ BAD,

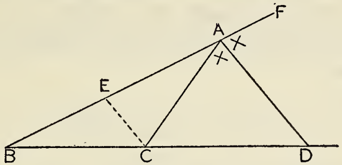
and $\angle$ ACE $= \angle$ DAC.

$\therefore \angle$ BAD $= \angle$ DAC,

and AD bisects $\angle$ BAC.

PROPOSITION 10 THEOREM

The bisector of the exterior vertical angle of a triangle divides the base externally into segments that are proportional to the sides of the triangle.



Hypothesis.—In $\triangle$ ABC, BA is produced to F.

$\angle$ FAC is bisected by AD which cuts BC produced at D.

Prove that $\frac{BD}{CD} = \frac{AB}{AC}$.

Construction.—Through C draw CE $\parallel$ AD to meet AB at E.

Proof.—$\because$ EC $\parallel$ AD, $\therefore$ $\angle$ FAD $=$ $\angle$ AEC.

and $\angle$ DAC $=$ $\angle$ ACE.

But, by hypothesis, $\angle$ FAD $=$ $\angle$ DAC.

$\therefore$ $\angle$ AEC $=$ $\angle$ ACE.

$\therefore$ AC $=$ AE. (I—12.)

In $\triangle$ BAD, EC $\parallel$ AD,

$\therefore \frac{BA}{AE} = \frac{BD}{DC}$. (IV—2, Cor.)

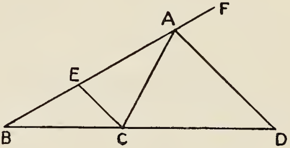
But AE $=$ AC.

$\therefore \frac{BA}{AC} = \frac{BD}{CD}$.

PROPOSITION 11 THEOREM

(Converse of IV—10.)

If the base of a triangle be divided externally so that the segments of the base are proportional to the other sides of the triangle, the straight line which joins the point of section to the vertex bisects the exterior vertical angle.



Hypothesis.—In $\triangle$ ABC, $\frac{BD}{CD} = \frac{BA}{AC}$, and BA is produced to F.

Prove that AD bisects $\angle$ CAF.

Construction.—Draw CE $\parallel$ AD to meet AB at E.

$\because \frac{BD}{DC} = \frac{BA}{AC}$, by hypothesis

and $\frac{BD}{DC} = \frac{BA}{AE}$. (IV—2 Cor.)

$\therefore$ AE = AC.

$\therefore \angle$ AEC = $\angle$ ACE. (I—2)

But $\angle$ AEC = $\angle$ FAD,

and $\angle$ ACE = $\angle$ CAD,

$\therefore \angle$ FAD = $\angle$ CAD.

and AD bisects $\angle$ CAF.

139.—Exercises

1. The sides of a △ are 4 cm., 5 cm., 6 cm. Calculate the lengths of the segments of each side made by the bisector of the opposite ∠.

2. **AD** bisects ∠ **A** of △ **ABC** and meets **BC** at **D.** Find **BD** and **CD** in terms of a, b, and c.

3. In △ **ABC,** $a = 7$, $b = 5$, $c = 3$. The bisectors of the exterior ∠s at **A, B, C** meet **BC, CA, AB** respectively at **D, E, F.** Calculate **BD, AE** and **AF.**

4. In △ **ABC,** the bisector of the exterior ∠ at **A** meets **BC** produced at **D.** Find **BD** and **CD** in terms of a, b and c.

5. If a st. line bisects both the vertical ∠ and the base of a △, the △ is isosceles.

6. The bisectors of the ∠s of a △ are concurrent. (Use IV—8 and 9.)

7. **AD** is a median of △ **ABC;** ∠s **ADB, ADC** are bisected by **DE, DF** meeting **AB, AC** at **E, F** respectively. Prove **EF** || **BC.**

8. The bisectors of ∠s **A, B, C** in △ **ABC** meet **BC, CA, AB** at **D, E, F** respectively. Show that **AF.BD.CE** = **FB.DC.EA.**

9. If the bisectors of ∠s **A, C** in the quadrilateral **ABCD** meet in the diagonal **BD,** the bisectors of ∠s **B, D** meet in the diagonal **AC.**

10. If the bisectors of ∠s **ABC, ADC** in the quadrilateral **ABCD** meet at a point in **AC,** the bisectors of the exterior ∠s at **B** and **D** meet in **AC** produced.

11. If **O** be the centre of the inscribed circle of △ **DEF** and **DO** produced meets **EF** at **G,** prove that **DO : OG** = **ED** + **DF : EF.**

12. **PQ** is a chord of a circle ⊥ to a diameter **MN** and **D** is any point in **PQ.** The st. lines **MD, ND** meet the circle at **E, F** respectively. Prove that any two adjacent sides of the quadrilateral **PEQF** are in the same ratio as the other two.

13. The bisector of the vertical ∠ of a △ and the bisectors of the exterior ∠s at the base are concurrent.

14. One circle touches another internally at **M**. A chord **PQ** of the outer circle touches the inner circle at **T**. Prove that $\frac{PT}{TQ} = \frac{PM}{MQ}$.

15. **LMN** is a $\triangle$ in which **LM** = 3 **LN**. The bisector of $\angle$ **L** meets **MN** in **D**, and **MX** $\perp$ **LD**. Prove that **LD** = **DX**.

16. The $\angle$ **A** of $\triangle$ **ABC** is bisected by **AD**, which cuts the base at **D**, and **O** is the middle point of **BC**. Show that **OD** is to **OB** as the difference of **AB** and **AC** is to their sum.

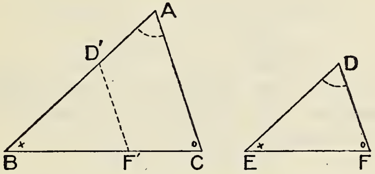
17. The bisectors of the interior and exterior $\angle$s at the vertex of a $\triangle$ divide the base internally and externally in the same ratio.

18. If **L**, **M**, **N** are three points in a st. line, and **P** a point at which **LM** and **MN** subtend equal $\angle$s, the locus of **P** is a circle.

Similar Triangles

Proposition 12 Theorem

If the angles of one triangle are respectively equal to the angles of another, the corresponding sides of the triangles are proportional.



Hypothesis.—In $\triangle$s ABC, DEF; $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E, $\angle$ C = $\angle$ F.

Prove that $$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

Proof.—Apply $\triangle$ DEF to $\triangle$ ABC so that $\angle$ E coincides with $\angle$ B; the $\triangle$ DEF taking the position D′BF′.

$\because \angle BD'F' = \angle A, \therefore D'F' \parallel AC.$ (Int. 41.)

$$\therefore \frac{AB}{D'B} = \frac{CB}{F'B} \qquad \text{(IV—2, Cor.)}$$

$$\therefore \frac{AB}{DE} = \frac{BC}{EF}.$$

In the same way, by applying the $\triangle$s so that $\angle$s C and F coincide, it may be proved that $\frac{BC}{EF} = \frac{CA}{FD}$.

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

Note.—$\because \frac{AB}{DE} = \frac{BC}{EF}$, $\therefore \frac{AB}{BC} = \frac{DE}{EF}$,

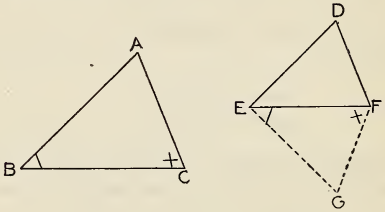
and in the same way $\frac{BC}{CA} = \frac{EF}{FD}$ and $\frac{CA}{AB} = \frac{FD}{DE}$.

$\therefore$ **If two triangles are similar, the corresponding sides about the equal angles are proportional.**

Proposition 13 Theorem

(*Converse of Proposition 12*)

If the sides of one triangle are proportional to the sides of another, the triangles are similar, the equal angles being opposite corresponding sides.



Hypothesis.—In $\triangle$s ABC, DEF; $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Prove that $\angle A = \angle D$, $\angle B = \angle DEF$, $\angle C = \angle DFE$.

Construction.—Make $\angle FEG = \angle B$, $\angle EFG = \angle C$.

Proof.—In $\triangle$s ABC, GEF $\begin{cases} \angle A = \angle G, \\ \angle B = \angle GEF, \\ \angle C = \angle EFG. \end{cases}$

$\therefore \triangle ABC \parallel\!| \triangle GEF$,

and $\frac{AB}{GE} = \frac{BC}{EF}$. (IV—12.)

But, by hypothesis, $\frac{AB}{DE} = \frac{BC}{EF}$.

$$\therefore \frac{AB}{GE} = \frac{AB}{DE}, \quad \therefore GE = DE.$$

Similarly it may be proved that $GF = DF$.

In $\triangle$s DEF, GEF $\begin{cases} DE = GE, \\ EF \text{ is common}, \\ FD = FG. \end{cases}$

$$\therefore \triangle DEF \equiv \triangle GEF. \qquad \text{(I—3.)}$$

$\therefore \angle DEF = \angle GEF = \angle B, \angle DFE = \angle GFE = \angle C.$

$\therefore$ remaining $\angle D$ = remaining $\angle A$.

140.—Exercises

1. The st. line joining the middle points of the sides of a $\triangle$ is || to the base, and equal to half of it.

2. If two sides of a quadrilateral are ||, the diagonals cut each other proportionally.

3. In the $\triangle$ ABC the medians BE, CF cut at G. Show that BG = twice GE, and CG = twice GF.

4. Using the theorem in Ex. 3, devise a method of trisecting a st. line.

5. If three st. lines meet at a point, they intercept on any || st. lines portions which are proportional to one another.

6. In similar $\triangle$s $\perp$s from corresponding vertices to the opposite sides are in the same ratio as the corresponding sides.

7. In similar $\triangle$s the bisectors of two corresponding $\angle$s, terminated by the opposite sides, are in the same ratio as the corresponding sides.

8. ABCD is a ||gm, and a line through A cuts BD at E, BC at F and meets DC produced at G. Show that AE : EF = AG : AF.

9. If two || st. lines AB, CD are divided at E, F respectively so that AE : EB = CF : FD, then AC, BD and EF are concurrent.

10. The median drawn to a side of a △ bisects all st. lines || to that side and terminated by the other two sides, or those sides produced.

11. **ABCD** is a ||gm. **AD** is bisected at **E** and **BC** at **F**. Show that **AF** and **CE** trisect the diagonal **BD**.

12. If the st. lines **OAB, OCD, OEF** are similarly divided, the △s **ACE, BDF** are similar.

13. If the corresponding sides of two similar △s are ||, the st. lines joining the corresponding vertices are concurrent.

14. △ **LMN** ||| △ **PQR**, ∠ **L** = ∠ **P** and ∠ **M** = ∠ **Q**. **LM** = 7 cm., **MN** = 5 cm., **LN** = 9 cm., **QR** = 4 cm. Find **PQ** and **PR**.

15. In △ **DEF, DE** = 13 cm., **EF** = 5 cm. and **DF** = 12 cm. The △ is folded so that the point **D** falls on the point **E**. Find the length of the crease.

16. **LMN** is a △ and **X** is any point in **MN**. Prove that the radii of the circles circumscribing **LMX, LNX** are proportional to **LM, LN**.

17. St. lines **POQ, ROS** are drawn so that **PO** = 2 **OQ** and **RO** = 2 **OS**. **RQ** and **PS** are produced to meet at **T**. Prove that **PS** = **ST** and **RQ** = **QT**.

18. **FDE, GDE** are two circles and **FDG** is a st. line. **FE, GE** are drawn. Prove that **FE** is to **GE** as diameter of circle **FDE** is to diameter of **GDE**.

19. **P** is any point on either arm of an ∠ **XOY**, and **PN** ⊥ to the other arm. Show that $\frac{PN}{OP}$ has the same value for all positions of **P**.

Show also that $\frac{ON}{OP}$ has the same value for all positions of **P**; and that $\frac{PN}{ON}$ has the same value for all positions of **P**.

(Note.—*The ratio* $\frac{PN}{OP}$ *is called the* sine *of the* ∠ **XOY**, $\frac{ON}{OP}$ *is the* cosine *of that* ∠, *and* $\frac{PN}{ON}$ *is the* tangent *of the same* ∠.)

20. **PQRS** is a quadrilateral inscribed in a circle. The diagonals **PR, QS** cut at X. Prove that $\frac{PQ}{SR} = \frac{XP}{XS}$.

21. **OX, OY, OZ** are three fixed st. lines, and **P** is any point in **OZ**. From **P, PL** is drawn ⊥ **OX** and **PM** ⊥ **OY**. Prove that the ratio **PL : PM** is constant.

22. In the quadrilateral **DEFG** the side **DE** || **GF** and the diagonals **DF, EG** cut at **H**. Through **H** the line **LHM** is drawn || **DE** and meeting **EF, DG** at **L, M** respectively. Prove **HL = HM.**

23. **KLMN** is a quadrilateral in which **KL** || **NM**. Prove that the line joining the middle points of **KL** and **MN** passes through the intersection of the diagonals **KM, LN.**

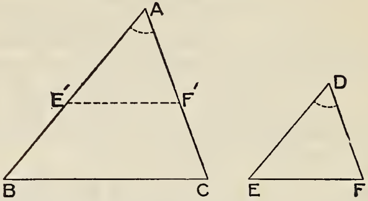
24. **DEF** is a △ and **G** is any point in **EF**. The bisector of ∠ **DGF** meets **DF** in **H**. **EH** cuts **DG** at **K**. **FK** meets **DE** at **L.** Prove that **LG** bisects ∠ **DGE.**

25. **DG** and **DH** bisect the interior and exterior ∠s at **D** of a △ **DEF**, and meet **EF** at **G** and **H**; and **O** is the middle point of **EF.** Show that **OE** is a mean proportional between **OG** and **OH.**

26. **DG** bisects ∠ **D** of △ **DEF** and meets **EF** at **G**. **GK** bisects ∠ **DGE** and meets **DE** at **K**. **GH** bisects ∠ **DGF** and meets **DF** at **H**. Prove that △ **EKH** : △ **FKH** = **ED** : **DF.**

PROPOSITION 14 THEOREM

If two triangles have one angle of one equal to one angle of the other and the sides about these angles proportional, the triangles are similar, the equal angles being opposite corresponding sides.



Hypothesis.—In $\triangle$s ABC, DEF, $\angle A = \angle D$

$$\text{and } \frac{AB}{DE} = \frac{AC}{DF}.$$

Prove that $\triangle ABC \,|||\, \triangle DEF$.

Proof.—Apply the $\triangle$s so that $\angle D$ coincides with $\angle A$ and $\triangle DEF$ takes the position $AE'F'$.

$$\because \frac{AB}{DE} = \frac{AC}{DF},$$

$$\therefore \frac{AB}{AE'} = \frac{AC}{AF'}, \quad \therefore E'F' \parallel BC. \qquad \text{(IV—3.)}$$

$$\therefore \angle B = \angle AE'F', \angle C = \angle AF'E',$$

$$\therefore \triangle ABC \,|||\, \triangle AE'F'.$$

But $\triangle AE'F'$ is the triangle DEF in its new position,

$$\therefore \triangle ABC \,|||\, \triangle DEF.$$

The equal $\angle$s B, E are respectively opposite the corresponding sides AC, DF, also the equal $\angle$s C, F are respectively opposite the corresponding sides AB, DE.

PROPOSITION 15 THEOREM

If two triangles have two sides of one proportional to two sides of the other, and the angles opposite one pair of corresponding sides in the proportion equal, the angles opposite the other pair of corresponding sides in the proportion are either equal or supplementary.

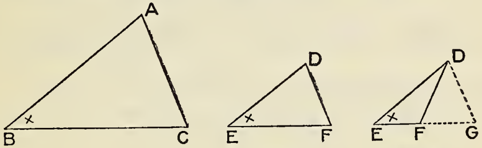


FIG. 1 FIG. 2

Hypothesis.—In $\triangle$s ABC, DEF, $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle B = \angle E$.

Prove that either $\angle C = \angle F$ or $\angle C + \angle DFE = 2$ rt. $\angle$s.

Proof.—The $\angle$s A and D are either equal or unequal.

(1) If $\angle A = \angle D$. (Fig. 1.)

$\because \angle A = \angle D$, and $\angle B = \angle E$, $\therefore \angle C = \angle F$.

In this case $\triangle ABC \parallel\!| \triangle DEF$.

(2) If $\angle A$ is not equal to $\angle D$. (Fig. 2.)

At D make $\angle EDG = \angle A$ and produce DG to meet EF, produced if necessary, at G.

$$\text{In } \triangle\text{s ABC, DEG} \left\{ \begin{array}{l} \angle A = \angle EDG, \\ \angle B = \angle E, \\ \therefore \angle C = \angle G. \end{array} \right.$$

$$\therefore \triangle ABC \parallel\!| \triangle DEG,$$

$$\text{and } \frac{AB}{DE} = \frac{AC}{DG}. \qquad \text{(IV—12.)}$$

But, by hypothesis, $\dfrac{AB}{DE} = \dfrac{AC}{DF}$.

$$\therefore \frac{AC}{DG} = \frac{AC}{DF}, \therefore DG = DF.$$

In $\triangle$ DFG, $\because$ DF = DG, $\therefore \angle$ DGF = $\angle$ DFG.

But $\angle$ DGF = $\angle$ C, $\therefore \angle$ DFG = $\angle$ C.

$\angle$ DFE + $\angle$ DFG = 2 rt. $\angle$s,

$\therefore \angle$ DFE + $\angle$ C = 2 rt. $\angle$s.

141.—Exercises

1. Show that certain propositions of Book I are respectively particular cases of Theorems 13, 14 and 15 of Book IV.

2. In similar $\triangle$s medians drawn from corresponding vertices are proportional to the corresponding sides.

3. In a $\triangle$ ABC, AD is drawn $\perp$ BC. If BD : DA = DA : DC, prove that BAC is a rt. $\angle$.

4. If the diagonals of a quadrilateral divide each other proportionally, one pair of sides are $\parallel$.

5. A point D is taken within a $\triangle$ LMN and joined to L and M. A $\triangle$ EMN is described on the other side of MN from $\triangle$ LMN having $\angle$ EMN = $\angle$ DML, and $\angle$ ENM = $\angle$ DLM. Prove that $\triangle$ DME $\parallel\!|$ $\triangle$ LMN.

6. M, N are fixed points on the circumference of a given circle, and P is any other point on the circumference. MP is produced to Q so that PQ : PN is a fixed ratio. Find the locus of Q.

7. **EOD, GOF** are two st. lines such that $GO : DO = EO : FO$. Prove that **E, F, D, G** are concyclic.

8. **OEF, OGD** are two st. lines such that $OE : OG = OD : OF$. Prove that **E, F, G, D** are concyclic.

9. **DEF** is a $\triangle$, and $FX \perp DE$. Prove that, if $DF : FX = DE : EF$, $\angle XFE = \angle D$.

10. Similar isosceles $\triangle$s **DEF, DEG** are described on opposite sides of **DE** such that $DF = DE$ and $GD = GE$. **H** is any point in **DF** and **K** is taken in **GD** such that $GK : GD = DH : DF$. Prove $\triangle KHE ||| \triangle GDE$.

11. **LMN** is a $\triangle$, and **D** is any point in **LM** produced. **E** is taken in **NM** such that $NE : EM = LD : DM$. Prove that **DE** produced bisects **LN**.

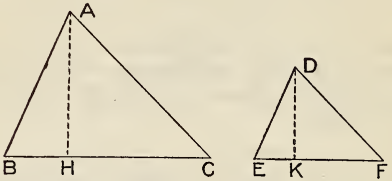
12. **O** is the centre and **OD** a radius of a circle. **E** is any point in **OD**, and **F** is taken in **OD** produced such that **OF** is a third proportional to **OE, OD**. **P** is any point on the circumference. Prove $\angle FPD = \angle DPE$.

13. The bisectors of the interior and exterior $\angle$s at **L** in the $\triangle$ **LMN** meet **MN** and **MN** produced at **D, E** respectively. **FNG** drawn $\parallel$ **LM** meets **LE** at **F** and **LD** produced at **G**. Prove $FN = NG$.

14. If one pair of $\angle$s of two $\triangle$s be equal and another pair of $\angle$s be supplementary, the ratios of the sides opposite to these pairs of $\angle$s are equal to each other.

PROPOSITION 16 THEOREM

The areas of similar triangles are proportional to the squares on corresponding sides.



Hypothesis.—ABC, DEF are similar $\triangle$s of which BC, EF are corresponding sides.

Prove that $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{BC^2}{EF^2} = \dfrac{AB^2}{DE^2} = \dfrac{AC^2}{DF^2}.$

Construction.—Draw $AH \perp BC$ and $DK \perp EF$.

Proof.— $\because \triangle AHC \mathbin{|||} \triangle DKF,$

$\therefore \dfrac{AH}{DK} = \dfrac{AC}{DF}.$ (IV—12.)

And $\because \triangle ABC \mathbin{|||} \triangle DEF,$

$\therefore \dfrac{AC}{DF} = \dfrac{BC}{EF}.$

$\therefore \dfrac{AH}{DK} = \dfrac{BC}{EF}.$

$\triangle ABC = \tfrac{1}{2} AH.BC,$ (II—13.)

$\triangle DEF \quad \tfrac{1}{2} DK.EF,$

$\therefore \dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\tfrac{1}{2} AH.BC}{\tfrac{1}{2} DK.EF}$

$$= \frac{AH}{DK} \cdot \frac{BC}{EF}$$

$$= \frac{BC}{EF} \cdot \frac{BC}{EF}$$

$$= \frac{BC^2}{EF^2}.$$

Similarly $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB^2}{DE^2} = \dfrac{AC^2}{DF^2}.$

142.—Exercises

1. Two similar $\triangle$s have corresponding sides in the ratio of 3 to 5. What is the ratio of their areas?

2. The ratio of the areas of two similar $\triangle$s equals the ratio of 64 to 169. What is the ratio of their corresponding sides?

3. Draw a $\triangle$ having sides 4 cm., 5 cm., 6 cm. Make a second $\triangle$ having its area four times that of the first, and divide it into parts each equal and similar to the first $\triangle$.

4. Show that the areas of similar $\triangle$s are as:—
   (*a*) the squares on corresponding altitudes;
   (*b*) the squares on corresponding medians;
   (*c*) the squares on the bisectors of corresponding $\angle$s.

5. **ABC**, **DEF** are two similar $\triangle$s such that area of $\triangle$ **DEF** is twice that of $\triangle$ **ABC**. What is the ratio of corresponding sides?

   Draw $\triangle$ **ABC** having sides 5 cm., 6 cm., 7 cm., and make $\triangle$ **DEF** similar to $\triangle$ **ABC**, and of double the area.

6. If **ABC**, **DEF** are similar $\triangle$s of which **BC**, **EF** are corresponding sides, and the st. line **G** be such that $BC : EF = EF : G$, then $\triangle ABC : \triangle DEF = BC : G$; that is:—

   If three st. lines are in continued proportion, the first is to the third as any $\triangle$ on the first is to the similar $\triangle$ similarly described on the second.

   NOTE.—*Similar $\triangle$s are said to be similarly described on corresponding sides.*

7. **ABC** is a $\triangle$ and **G** is a st. line. Describe a $\triangle$ **DEF** similar to $\triangle$ **ABC** and such that $\triangle$ **ABC** : $\triangle$ **DEF** = **BC** : **G**.

Describe another $\triangle$ **HKL** similar to $\triangle$ **ABC** and such that $\triangle$ **ABC** : $\triangle$ **HKL** = **AB** : **G**.

8. Bisect a given $\triangle$ by a st. line drawn || to one of its sides.

9. From a given $\triangle$ cut off a part equal to one-third of its area by a st. line drawn || to one of its sides.

10. Trisect a given $\triangle$ by st. lines drawn || to one of its sides.

11. Show that the equilateral $\triangle$ described on the hypotenuse of a rt.-$\angle$d $\triangle$ equals the sum of the equilateral $\triangle$s on the two sides.

12. In $\triangle$ **DEF**, **DX** $\perp$ **EF** and **EY** $\perp$ **FD**. Prove that $\triangle$ **FXY** : $\triangle$ **DEF** = $FX^2 : FD^2$.

13. In the acute-$\angle$d $\triangle$ **DEF**, **DX** $\perp$ **EF**, **EY** $\perp$ **FD**, **FZ** $\perp$ **DE**, **YG** $\perp$ **EF** and **ZH** $\perp$ **EF**. Prove that **XY** and **XZ** divide the $\triangle$ **DEF** into three parts that are proportional to **FG**, **GH** and **HE**.

14. **LMN** is an equilateral $\triangle$. The st. lines **RLQ**, **PMR**, **QNP** are respectively $\perp$ **LM**, **MN**, **NL**. Find the ratio of $\triangle$ **PQR** to $\triangle$ **LMN**.

15. A point **O** is taken in the diameter **PQ** produced of a circle. **OT** is a tangent and the tangent at **P** cuts **OT** at **N**. If **D** is the centre of the circle, prove that $\triangle$ **OPN** : $\triangle$ **OTD** = **OP** : **OQ**.

16. **H** is a point on the circumference of a circle of which **FG** is a diameter, and **O** is the centre. **HD** $\perp$ **FG**, and tangents at **F** and **H** meet at **E**. Prove that $\triangle$ **FEH** : $\triangle$ **OHG** = **FD** : **DG**.

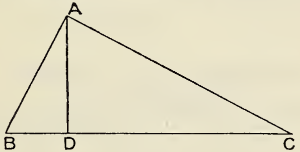
17. **DEF**, **LMN** are two $\triangle$s in which $\angle$ **E** = $\angle$ **M**. Prove that $\triangle$ **DEF** : $\triangle$ **LMN** = **DE.EF** : **LM.MN**.

18. Similar $\triangle$s are to one another as the squares on the radii of their circumscribing circles.

Geometric Means

Proposition 17 Theorem

The perpendicular from the right angle to the hypotenuse in a right-angled triangle divides the triangle into two triangles which are similar to each other and to the original triangle.



Hypothesis.—In $\triangle$ ABC, $\angle$ BAC is a rt. $\angle$ and AD $\perp$ BC.

Prove that $\triangle$ ABD $|||$ $\triangle$ CAD $|||$ $\triangle$ CBA.

Proof.—

In $\triangle$s ABD, CBA $\begin{cases} \angle \text{ B is common.} \\ \angle \text{ BDA} = \angle \text{ BAC, both rt. } \angle\text{s.} \\ \therefore \angle \text{ BAD} = \angle \text{ BCA.} \end{cases}$

$\therefore \triangle$ ABD $|||$ $\triangle$ CBA

Similarly $\triangle$ ADC $|||$ $\triangle$ CBA.

$\therefore \triangle$ ABD $|||$ $\triangle$ CAD $|||$ $\triangle$ CBA.

Cor. 1.— $\because \triangle$ ABD $|||$ $\triangle$ CAD,

$$\therefore \quad \frac{BD}{AD} = \frac{AD}{CD}.$$

$\therefore$ AD is the mean proportional between BD and DC.

Cor. 2.—Because $\triangle$ ABD $|||$ $\triangle$ CBA,

$$\therefore \quad \frac{BD}{AB} = \frac{AB}{BC}.$$

∴ AB is the mean proportional between BD and BC.

Similarly—

AC is the mean proportional between DC and CB.

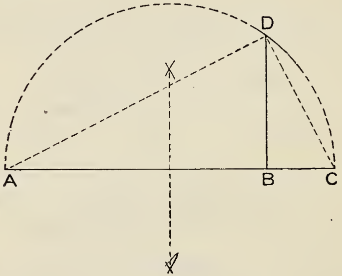
Cor. 3.—Because $\triangle$ CBA ||| $\triangle$ CAD,

$$\therefore \quad \frac{CB}{BA} = \frac{CA}{AD}.$$

i.e., the hypotenuse is to one side as the other side is to the perpendicular.

Proposition 18 Problem

Find the mean proportional between two given straight lines.



From a st. line cut off AB, BC respectively equal to the two given st. lines.

It is required to find the mean proportional to AB, BC.

On AC as diameter describe a semi-circle ADC. From B draw BD $\perp$ AC and meeting the arc ADC at D.

BD is the required mean proportional.

Join AD, DC.

Proof.— $\because$ ADC is a semi-circle,

$\therefore \angle$ ADC is a rt. $\angle$. (III—11.)

In $\triangle$ ADC, $\angle$ ADC is a rt. $\angle$,

and DB $\perp$ AC.

$\therefore \frac{AB}{BD} = \frac{DB}{BC}$. (IV—17, Cor. 1)

$\therefore$ BD is the mean proportional between AB and BC.

143.—Exercises

1. Give a general enunciation of IV—17, Cor. 1.

2. Give a general enunciation of IV—17, Cor. 2.

3. In any two equal $\triangle$s ABC, DEF, if AG, DH be $\perp$s to BC, EF respectively, AG : DH = EF : BC.

4. In the diagram of IV—17, show that rect. AD.BC = rect. BA.AC. Give a general statement of this theorem.

5. ABC, DEF are two equal $\triangle$s having also $\angle$ B = $\angle$ E. Show that $\frac{BC}{EF} = \frac{DE}{AB}$.

6. ABCD, EFGH are two equal ||gms having also $\angle$ B = $\angle$ F. Show that $\frac{BC}{FG} = \frac{FE}{BA}$.

7. ABCD is a given rect. and EF a given st. line. It is required to make a rect. equal in area to ABCD and having one of its sides equal to EF.

8. Make a rect. equal in area to a given $\triangle$ and having one of its sides equal to a given st. line.

9. Show how to construct a rect. equal in area to a given polygon and having one of its sides equal to a given st. line.

10. If from any point on the circumference of a circle a $\perp$ be drawn to a diameter, the square on the $\perp$ equals the rect. contained by the segments of the diameter.

11. Construct a square equal to a given rect.

12. Construct a square equal to a given ||gm.

13. Construct a square equal to a given $\triangle$.

14. Draw a square having its area 12 sq. inches.

15. Divide a given st. line into two parts such that the rect. contained by the parts is equal to the square on another given st. line.

16. If a st. line be divided into two parts, the rect. contained by the parts is greatest when the line is bisected.

17. **AB** and **C** are two given st. lines. Find a point **D** in **AB** produced such that rect. **AD. DB** = sq. on **C.**

18. Construct a rect. equal in area to a given square and having its perimeter equal to a given st. line.

When will the solution be impossible?

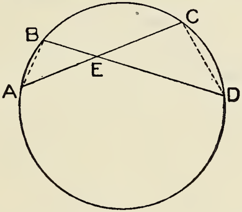
19. Show how to construct a square equal in area to a given polygon.

20. In the corresponding sides **BC, EF** of the similar $\triangle$s **ABC, DEF** the points **G, H** are taken such that **BG : GC = EH : HF.** Prove **AG : DH = BC : EF.**

CHORDS AND TANGENTS

PROPOSITION 19 THEOREM

If two chords intersect within a circle, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.



Hypothesis.—In the circle ABC, the chords AC, BD intersect at E.

Prove that rect. AE.EC = rect. BE.ED.

Construction.—Join AB, CD.

Proof.—$\because$ $\angle$s ABD, ACD are in the same segment,

$\therefore$ $\angle$ ABD = $\angle$ ACD. (III—9.)

Similarly, $\angle$ BAC = $\angle$ BDC.

And $\angle$ AEB = $\angle$ CED.

$\therefore$ $\triangle$ AEB ||| $\triangle$ DCE.

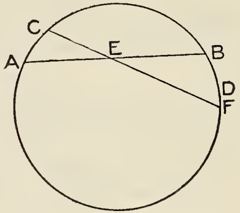
$$\therefore \quad \frac{AE}{ED} = \frac{BE}{EC}. \qquad \text{(IV—12.)}$$

$\therefore$ rect. AE.EC = rect. BE.ED.

Proposition 20 Theorem

(*Converse of IV—19*)

If two straight lines cut each other so that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other, the four extremities of the two straight lines are concyclic.



Hypothesis.—The st. lines AB, CD cut at E so that rect. AE.EB = rect. CE.ED.

Prove that A, C, B, D are concyclic.

Construction.—Describe a circle through A, C, B, and let it cut ED, produced if necessary, at F.

Proof.— $\because$ AB, CF are chords of a circle,

$\therefore$ AE.EB = CE.EF. (IV—19.)

But, AE.EB = CE.ED. (*Hyp.*)

$\therefore$ CE.EF = CE.ED.

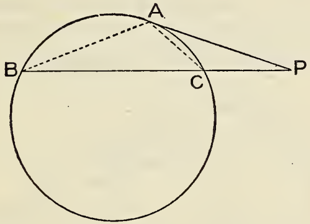
And $\therefore$ EF = ED.

$\therefore$ F coincides with D,

and the points A, C, B, D are concyclic.

Proposition 21 Theorem

If from a point without a circle a secant and a tangent are drawn, the square on the tangent is equal to the rectangle contained by the secant and the part of it without the circle.



Hypothesis.—PA is a tangent and PCB a secant to the circle ABC.

Prove that $PA^2 = PB.PC$.

Construction.—Join AB, AC.

Proof.—∵ AP is a tangent, and AC is a chord from the same point A,

$$\therefore \angle PAC = \angle ABC. \qquad \text{(III—18.)}$$

$$\text{In } \triangle\text{s PAB, PCA, } \begin{cases} \angle P \text{ is common,} \\ \angle PBA = \angle PAC, \\ \text{and } \therefore, \angle PAB = \angle PCA, \end{cases}$$

$$\therefore \triangle PAB \mid\mid\mid \triangle PCA.$$

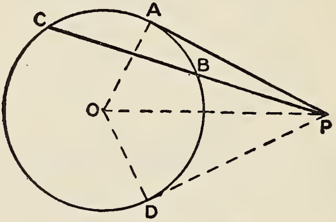
$$\therefore \frac{PB}{PA} = \frac{PA}{PC}. \qquad \text{(IV—12.)}$$

$$\therefore PA^2 = PB.PC.$$

Proposition 22 Theorem

(*Converse of IV—21*)

If from a point without a circle two straight lines are drawn, one of which is a secant and the other meets the circle so that the square on the line which meets the circle is equal to the rectangle contained by the secant and the part of it without the circle, the line which meets the circle is a tangent.



Hypothesis.—PA and PBC are drawn to the circle ABC so that $PA^2 = PB.PC$.

Prove that PA is a tangent.

Construction.—Find the centre O. From P draw a tangent PD to the circle. Join OA, OP, OD.

Proof.— $\because$ PD is a tangent,

$\therefore PD^2 = PB.PC.$ (IV—21.)

But $PA^2 = PB.PC.$ (*Hyp.*)

$\therefore PD^2 = PA^2,$

and $\therefore PD = PA.$

In $\triangle$s POA, POD, $\begin{cases} OA = OD, \\ AP = DP, \\ OP \text{ is common.} \end{cases}$

$\therefore \angle OAP = \angle ODP.$ (I—3.)

$= \text{a rt. } \angle.$ (III—16.)

Then $\because$ PA is $\perp$ the radius OA,

$\therefore$ PA is a tangent. (III—16, Cor. 1.)

144.—Exercises

1. PAB, PCD are two secants drawn from a point P without a circle. Show that rect. PA.PB = rect. PC.PD.

From this exercise deduce a proof for IV—21.

2. If in two st. lines PB, PD points A, C respectively are taken such that rect. PA.PB = rect. PC.PD, the four points A, B, C, D are concyclic.

3. If two circles intersect, their common chord bisects their common tangents.

4. If two circles intersect, the tangents drawn to them from any point in their common chord produced are equal to each other.

5. Through P any point in the common chord, or the common chord produced, of two intersecting circles two lines are drawn cutting one circle at A, B and the other at C, D. Show that A, B, C, D are concyclic.

6. Through a point P within a circle, any chord APB is drawn. If O be the centre, show that rect. AP.PB = $OA^2 - OP^2$.

7. From any point P without a circle any secant PAB is drawn. If O be the centre, show that rect. PA.PB = $OP^2 - OA^2$.

8. From a given point as centre describe a circle cutting a given st. line in two points, so that the rectangle contained by their distances from a given point in the st. line may be equal to a given square.

9. Describe a circle to pass through two given points and touch a given st. line.

10. If three circles are drawn so that each intersects the other two, the common chords of each pair meet at a point.

11. Find a point **D**, in the side **BC** of $\triangle$ **ABC**, such that the sq. on **AD** = rect. **BD.DC**. When is the solution possible?

12. Use IV—21 to find a mean proportional to two given st. lines.

13. **P** is a point at a distance of 7 cm. from the centre of a circle. **PDE** is a secant such that **PD** = 5 cm. and **DE** = 3 cm. Find the length of the radius of the circle.

14. In a circle of radius 4 cm. a chord **DE** is drawn 7 cm. in length. **F** is a point in **DE** such that **DF** = 5 cm. Find the distance of **F** from the centre of the circle.

15. **DEF** is an isosceles $\triangle$ in which **ED** = **EF**. A circle, which passes through **D** and touches **EF** at its middle point cuts **DE** at **H**. Prove that **DH** = 3 **HE**.

16. In a circle two chords **DE**, **FG** cut at **H**. Prove that

$$(FH - HG)^2 - (DH - HE)^2 = FG^2 - DE^2.$$

17. **LND**, **MNE** are two chords intersecting inside a circle and **LM** is a diameter. Prove that

$$LN.LD + MN.ME = LM^2.$$

18. **DEF**, **HGF** are two circles and **DFG** is a fixed st. line. Show how to draw a st. line **EFH** such that **EF.FH** = **DF.FG**.

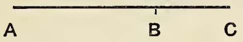
19. **P** is a point in the diameter **DE** of a circle, and **PT** is the $\perp$ on the tangent at a point **Q**. Prove that $PT.DE = DP.PE + PQ^2$.

20. **P**, **Q**, **R**, **S** are four points in order in the same st. line. Find a point **O** in this st. line such that **OP.OR** = **OQ.OS**.

21. The tangent at **P** to a circle, whose centre is **O** meets two $\parallel$ tangents in **Q**, **R**. Prove that $PQ.PR = OP^2$.

PROPOSITION 23 THEOREM

The square on the sum of two straight lines equals the sum of the squares on the two straight lines increased by twice the rectangle contained by the straight lines.



Hypothesis.—AB, BC are the two st. lines placed in the same st. line so that AC is their sum.

Prove that

$$AC^2 = AB^2 + BC^2 + 2.AB.BC.$$

Algebraic Proof

Proof.—Let a, b represent the number of units of length in AB, BC respectively.

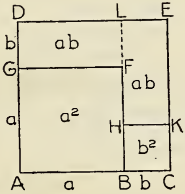
Area of the square on AC

$= (a + b)^2$

$= a^2 + b^2 + 2\,ab$

= area of square on AB + area of square on BC + twice the area of the rectangle contained by AB, and BC.

Geometric Proof



Construction.—On AC, AB, BC draw squares ACED, ABFG, BCKH. Produce BF to meet DE at L.

Proof.—

$GD = AD - AG = AC - AB = BC$, and $GF = AB$.

$\therefore GL =$ rect. $AB.BC$.

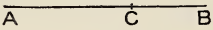
$KE = CE - CK = AC - BC = AB$, and $HK = BC$.

$\therefore HE =$ rect. $AB.BC$.

$$\begin{aligned} AC^2 &= AE \\ &= AF + BK + GL + HE \\ &= AB^2 + BC^2 + 2\,AB.BC. \end{aligned}$$

Proposition 24 Theorem

The square on the difference of two straight lines equals the sum of the squares on the two straight lines diminished by twice the rectangle contained by the straight lines.



Hypothesis.—AB, BC are two st. lines, of which AB is the greater, placed in the same st. line, and so that AC is their difference.

Prove that

$$AC^2 = AB^2 + BC^2 - 2\,.\,AB.BC.$$

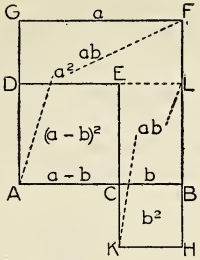
Algebraic Proof

Proof.—Let a, b represent the number of units of length in AB, BC respectively.

$$\begin{aligned} \text{Area of square on } AC &= (a - b)^2 \\ &= a^2 + b^2 - 2ab \end{aligned}$$

= the sum of the squares on AB and BC diminished by twice the area of the rectangle contained by AB and BC.

Geometric Proof



Construction.—On AC, AB, BC draw the squares ACED, ABFG, BCKH. Produce DE to meet BF at L.

Proof.—$DG = AG - AD = AB - AC = BC$, and $DL = AB$.

$\therefore DF =$ rect. AB . BC.

$KE = KC + CE = BC + AC = AB$, and $KH = BC$.

$\therefore KL =$ rect. AB . BC.

$$AC^2 = AE$$
$$= AF + KB - (DF + KL)$$
$$= AB^2 + BC^2 - 2\,AB \cdot BC.$$

145.—Exercises

1. The difference of the squares on two straight lines equals the rectangle of which the length is the sum of the straight lines and the breadth is the difference of the straight lines.

Prove this theorem by both the geometric and the algebraic methods.

2. The square on the sum of three st. lines equals the sum of the squares on the three st. lines increased by twice the sum of the rectangles contained by each pair of the st. lines.

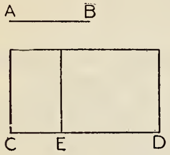
Illustrate by diagram.

3. The sum of the squares on two unequal st. lines $>$ twice the rectangle contained by the two st. lines.

4. The sum of the squares on three unequal st. lines $>$ the sum of the rectangles contained by each pair of the st. lines.

5. Construct a rectangle equal to the difference of two given squares.

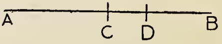
6. If there are two st. lines **AB** and **CD**, and **CD** be divided at **E** into any two parts, the rect. **AB . CD** = rect. **AB . CE** + rect. **AB . ED.**



Let **AB** $= p$ units of length.
CE $= q$ " " "
ED $= r$ " " "
Area of **AB . CD** $= p(q + r)$
" " **AB . CE** $= pq$
" " **AB . ED** $= pr$.
But $p(q + r) = pq + pr$.
$\therefore$ **AB . CD = AB . CE + AB . ED.**

7. Give a diagram illustrating the identity $(a + b)(c + d) = ac + ad + bc + bd$, taking a, b, c, d to be respectively the number of units in four st. lines.

8. **C** is the middle point of a st. line **AB**, and **D** is any other point in the line. Prove:

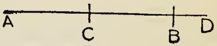


(1) $AD \cdot DB = AC^2 - CD^2$;

(2) $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

(Let $AC = CB = p$, $CD = q$).

9. **C** is the middle point of a st. line **AB**, and **D** is any point in **AB** produced. Prove:



(1) $AD \cdot DB = CD^2 - AC^2$;

(2) $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.

10. Draw diagrams to illustrate the four results in exercises 8 and 9.

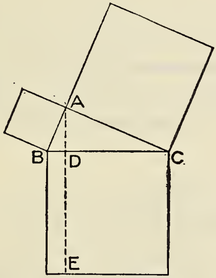
11. Draw a diagram illustrating the identity $(a + b)^2 - (a - b)^2 = 4\,ab$.

PROPOSITION 25 THEOREM

Alternative proof of the Pythagorean Theorem.

(*II—23.*)

The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides.



Hypothesis.—BAC is a $\triangle$ having $\angle$ BAC a rt. $\angle$, and having squares described on the three sides.

Prove that $BC^2 = BA^2 + AC^2$.

Construction.—Draw $AD \perp BC$.

Proof.—$\because$ BAC is a rt.-$\angle$d $\triangle$ with $AD \perp$ the hypotenuse BC,

$$\therefore \quad \frac{BC}{BA} = \frac{BA}{BD}. \qquad \text{(IV—17, Cor. 2.)}$$

$$\therefore \quad BA^2 = BC.BD.$$

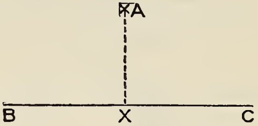
$$\text{Similarly} \quad CA^2 = BC.CD.$$

$$\begin{aligned} \therefore\ BA^2 + CA^2 &= BC.BD + BC.CD \\ &= BC\,(BD + CD) \\ &= BC.BC \\ &= BC^2 \end{aligned}$$

i.e., $BC^2 = BA^2 + CA^2$.

146. **Definition.**—If a perpendicular be drawn from a given point to a given straight line, the foot of the perpendicular is said to be the **projection of the point on the line.**

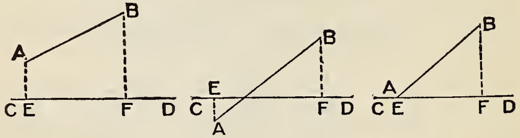
From the point A the ⊥ AX is drawn to the line BC.



The point X is the projection of the point A on the st. line BC.

147. **Definition.**—If from the ends of a given straight line perpendiculars be drawn to another given straight line, the segment intercepted on the second straight line is called the **projection of the first straight line on the second straight line.**

AB is a st. line of fixed length and CD another st. line. AE, BF are drawn ⊥ CD.



EF is the projection of AB on CD.

148.—Exercises

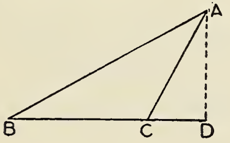
1. Show that a st. line of fixed length is never less than its projection on another st. line. In what case are they equal? In what case is the projection of one st. line on another st. line just a point?

2. ABC is a △ having $a = 36$ mm., $b = 40$ mm. and $c = 45$ mm. Draw the △ and measure the projection of AB on BC. (*Ans.* 23·9 mm. nearly.)

3. ABC is a △ having $a = 5$ cm., $b = 7$ cm., $c = 10$ cm. Draw the △ and measure the projection of AB on BC. (*Ans.* 76 mm.)

Proposition 26 Theorem

In an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides that contain the obtuse angle increased by twice the rectangle contained by either of these sides and the projection on that side of the other.



Hypothesis.—ABC is a △ in which ∠ C is obtuse, and CD is the projection of CA on CB.

To prove that $AB^2 = AC^2 + BC^2 + 2\,BC.CD$.

Proof.—∵ ADB is a rt. ∠,

∴ $AB^2 = BD^2 + AD^2$. (II—23.)

∵ $BD = BC + CD$,

∴ $BD^2 = BC^2 + CD^2 + 2\,BC.CD$. (IV—23.)

∴ $AB^2 = BC^2 + CD^2 + 2\,BC.\,CD + AD^2$.

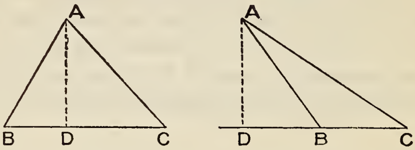
But ∵ ADC is a rt. ∠,

∴ $CD^2 + AD^2 = AC^2$.

∴ $AB^2 = AC^2 + BC^2 + 2\,BC.CD$.

PROPOSITION 27 THEOREM

In any triangle, the square on the side opposite an acute angle is equal to the sum of the squares on the sides which contain the acute angle diminished by twice the rectangle contained by either of these sides and the projection on that side of the other.



Hypothesis.—ABC is a $\triangle$ in which $\angle$ C is acute, and CD is the projection of CA on CB.

To prove that $AB^2 = AC^2 + BC^2 - 2\,BC.CD$.

Proof.—$\because$ ADB is a rt. $\angle$,

$\therefore AB^2 = BD^2 + AD^2$. (II—13.)

$\because$ BD is the difference between BC and CD,

$\therefore BD^2 = CD^2 + BC^2 - 2\,BC.CD$. (IV—24.)

$\therefore AB^2 = CD^2 + BC^2 - 2\,BC.CD + AD^2$.

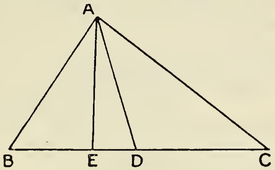
But, $\because$ ADC is a rt. $\angle$,

$\therefore CD^2 + AD^2 = AC^2$.

$\therefore AB^2 = AC^2 + BC^2 - 2\,BC.CD$.

Proposition 28 Theorem

In any triangle the sum of the squares on two sides equal twice the square on half the base together with twice the square on the median to the base.



Hypothesis.—ABC is a △ in which AD is a median.

Prove that $AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$.

Construction.—Draw AE ⊥ BC.

Proof.—$AB^2 = AD^2 + BD^2 - 2\,BD.ED.$ (IV—26.)

$AC^2 = AD^2 + DC^2 + 2\,DC.ED.$ (IV—27.)

$\therefore AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2$ since $DC = BD$.

149.—Exercises

1. In any ||gm the sum of the squares on the sides is equal to the sum of the squares on its diagonals.

2. ABC is a △, CD the projection of CA on CB, and CE the projection of CB on CA. Show that rect. BC.CD = rect. AC.CE.

3. In any quadrilateral the sum of the squares on the four sides exceeds the sum of the squares on the diagonals by four times the square on the st. line joining the middle points of the diagonals.

What does this proposition become when the quadrilateral is a ||gm?

4. **ABC** is a $\triangle$ having $a = 47$ mm., $b = 62$ mm., and $c = 84$ mm. **D, E, F** are the middle points of **BC, CA, AB** respectively. Calculate the lengths of **AD, BE** and **CF**. Test your results by drawing and measurement.

5. The squares on the diagonals of a quadrilateral are together double the sum of the squares on the st. lines joining the middle points of opposite sides.

6. If the medians of a $\triangle$ intersect at **G**,

$$AB^2 + BC^2 + CA^2 = 3\,(GA^2 + GB^2 + GC^2).$$

7. **C** is the middle point of a st. line **AB**. **P** is any point on the circumference of a circle of which **C** is the centre. Show that $PA^2 + PB^2$ is constant.

8. Two circles have the same centre. Prove that the sum of the squares of thé distances from any point on the circumference of either circle to the ends of the diameter of the other is constant.

9. The square on the base of an isosceles $\triangle$ is equal to twice the rect. contained by either of the equal sides and the projection on it of the base.

10. If, from any point **P** within $\triangle$ **ABC**, $\perp$s **PX, PY, PZ** be drawn to **BC, CA, AB** respectively,

$$BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2.$$

11. **D, E, F** are the middle points of **BC, CA, AB** respectively in $\triangle$ **ABC**. Prove that

$$3\,(AB^2 + BC^2 + CA^2) = 4\,(AD^2 + BE^2 + CF^2).$$

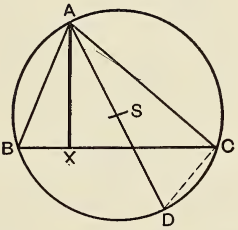
12. **G** is the centroid of $\triangle$ **ABC**, and **P** is any point. Show that

$$PA^2 + PB^2 + PC^2 = AG^2 + BG^2 + CG^2 + 3\,PG^2.$$

Areas of Rectangles

Proposition 29 Theorem

If from the vertex of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circumcircle of the triangle.



Hypothesis.—AX $\perp$ BC and AD is a diameter of the circumcircle of $\triangle$ ABC.

Prove that rect. AB.AC = rect. AX.AD.

Construction.—Join DC.

$$\because \angle AXB = \angle ACD,$$

$$\text{and } \angle ABX = \angle ADC.$$

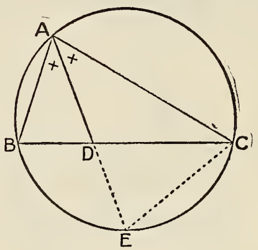
$$\therefore \triangle AXB \parallel\!| \triangle ACD.$$

$$\therefore \quad \frac{AB}{AD} = \frac{AX}{AC}.$$

$$\therefore \text{rect. } AB \,.\, AC = \text{rect. } AX \,.\, AD.$$

Proposition 30 Theorem

If the vertical angle of a triangle is bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the straight line which bisects the angle.



Hypothesis.—ABC is a $\triangle$ and AD the bisector of $\angle$ A.

Prove that the rect. AB . AC = rect. BD . DC + AD^2.

Construction.—Circumscribe a circle about the $\triangle$ ABC. Produce AD to cut the circumference at E. Join EC.

Proof.—

In $\triangle$s BAD, EAC $\angle$ BAD = $\angle$ EAC, $\angle$ ABD = $\angle$ AEC,

$\therefore$ $\angle$ ADB = $\angle$ ACE and the $\triangle$s are similar;

$$\text{hence} \qquad \frac{BA}{AD} = \frac{EA}{AC},$$

$$\text{and } \therefore \ BA \,.\, AC = AD \,.\, EA.$$

But $AD \,.\, EA = AD\,(AD + DE)$

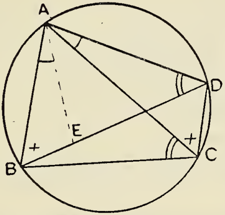
$$= AD^2 + AD \,.\, DE$$

$$= AD^2 + BD \,.\, DC.$$

$\therefore$ rect. BA . AC = rect. BD . DC + AD^2.

PROPOSITION 31 THEOREM

Ptolemy's Theorem.—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by its opposite sides.



Hypothesis.—ABCD is a quadrilateral inscribed in a circle.

Prove that AC . BD = AB . CD + BC . AD.

Construction.—Make $\angle$ BAE = $\angle$ CAD, and produce AE to cut BD at E,

Proof.— $\because \triangle ABE \parallel\!| \triangle ACD$,

$$\therefore \frac{AB}{AC} = \frac{BE}{CD}.$$

$$\therefore AB . CD = AC . BE.$$

$$\because \triangle ADE \parallel\!| \triangle ABC,$$

$$\therefore \frac{AD}{AC} = \frac{DE}{BC}$$

$$\therefore AD . BC = AC . DE.$$

$$\therefore AB . CD + AD . BC = AC . BE + AC . DE$$
$$= AC (BE + ED)$$
$$= AC . BD.$$

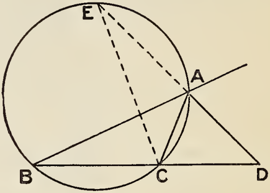
(HISTORICAL NOTE — Ptolemy, a native of Egypt, flourished in Alexandria in 139 A.D.)

150.—Exercises

(a)

1. If the exterior vertical ∠ A of △ ABC be bisected by a line which cuts BC produced at D, rect. AB . AC = rect. BD . CD − AD^2.

(Note.—Draw the circle ACB. Produce DA to cut the circumference at E. Draw EC. Prove △ BAD ||| △ EAC, and that ∴ $\frac{BA}{AD} = \frac{EA}{AC}$. Then BA . AC = AD . EA = AD (ED − AD) = AD . ED − AD^2 = BD . CD − AD^2.)



2. Draw △ ABC having a = 81 mm., b = 60 mm., c = 30 mm. Bisect the interior and exterior ∠s at A and produce the bisectors to meet BC and BC produced at D and E. Measure AD, AE; and check your results by calculation.

3. If the internal and external bisectors of ∠ A of △ ABC meet BC at L, M respectively, prove

$$LM^2 = BM . MC - BL . LC.$$

4. If R is the radius of the circumcircle and △ the area of △ ABC, prove that

$$R = \frac{abc}{4\triangle}.$$

If the sides of a △ are 39, 42, 45, show that R = 24⅜.

5. P is any point on the circumcircle of an equilateral △ ABC. Show that, of the three distances PA, PB, PC, one is the sum of the other two.

(b)

6. From any point P on a circle ⊥s are drawn to the four sides and to the diagonals of an inscribed quadrilateral. Prove that the rect. contained by the ⊥s on either pair of opposite sides is equal to the rect. contained by the ⊥s on the diagonals.

7. If the sides of an isosceles △ are 10 inches and the base 12 inches, find the radius of the circle circumscribed about it.

8. **ABC** is an isosceles △ and on the base **BC**, or base produced, a point **D** is taken; show that the circles about the triangles **ABD** and **ACD** are equal.

9. With given base and vertical ∠ construct a △ having the rect. contained by its sides equal to the square on a given st. line.

10. **A, B, C, D** are given points on a circle. Find a point **P** on the circle such that **PA . PC = PB . PD.**

11. **AB** is the chord of contact of tangents drawn from a point **P** to a circle. **PCD** cuts the circle at **C, D.** Prove that **AB . CD** = 2 **AC . BD.**

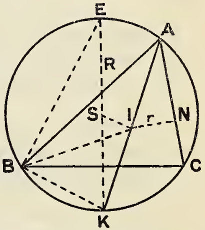
12. **I** is the centre of the inscribed circle of △ **ABC.** **AI** produced meets the circumcircle at **K.** Prove **AI . IK** = 2 **R***r*.

(NOTE.—Draw the diameter **KE** of circle **ABC** and the radius **IN** of the inscribed circle. Draw **BE, BK, BI.**

Show that △ **BEK** ||| △ **NAI**, and that $\therefore \dfrac{AI}{IN} = \dfrac{KE}{KB}$;

or, **AI . KB** = 2**R***r*.

Show that **KB = KI**, and that ∴ **AI. IK** = 2**R***r*.)



13. I_1 is the centre of the escribed circle opposite to **A** in △ **ABC.** AI_1 cuts the circumcircle **ABC** at **K.** Prove $AI_1 . I_1K = 2Rr_1$.

Hence show that, if **S** be the circumcentre of △ **ABC**, $SI_1^2 = R^2 + 2Rr_1$.

Miscellaneous Exercises

1. **EFGH** is a ||gm, **P** a point in **EF** such that **EP : PF** = $m:n$. What fraction is △ **EPH** of the ||gm?

2. **EFGH** is a ||gm, **P** is a point in the diagonal **FH** such that **FP : PH** = 2 : 5. What fraction of the ||gm is △ **EFP**? If **FP : PH** = $m:n$ find the fraction.

3. **EFGH** is a ||gm, **P** is a point in the diagonal **FH** produced such that **FP : PH** = 9 : 5. What fraction of the ||gm is the △ **PEH**?

4. **KLMN** is a ||gm. Any st. line **EKG** is drawn cutting the sides **ML** and **MN** produced at **E** and **G**. Show that half the ||gm is a mean proportional between △s **EKL** and **NKG**

5. The △ **PQR** has **PQ** and **QR** divided at **D** and **E** such that **PD : DQ** = **QE : ER** = 1 : 3. **PE** and **RD** intersect at **O**. Find the ratios of the △s **PDO : OPR : OER : PQR.**

6. **D** and **E** are points in **PQ** and **PR** sides of the △ **PQR** such that **QD : DP** = **PE : ER** = $m:n$. Compare the areas of the △s **QDE** and **DER.**

7. Either of the complements of the ||gms about the diagonal of a ||gm is a mean proportional between the two ||gms about the diagonal.

8. **LMN** is an isosceles △ having **LM** = **LN, LD** is perpendicular to **MN, P** is a point in **LN** such that **LP : LM** = 1 : 3. Prove that **MP** bisects **LD.**

9. Through **E** one of the vertices of a rectangle **EFGH** any st. line is drawn, and **HP** and **FQ** are ⊥s to **PEQ.** Prove **PE.EQ** = **HP.FQ.**

10. **DEF** is a △, **P** and **Q** are points in **DE** and **DF,** and **DP : PE** = 3 : 5 and **DQ : QF** = 7 : 8. In what ratio is **PQ** cut by the median **DG**?

11. DEFG is a ||gm, and EF is produced to K so that FK = EF; DK cuts EG at P. Show that GP = $\frac{1}{3}$ EG.

12. The diagonals of the ||gm EFGH intersect at O; if E be joined to the middle point P of OH, and EP and FG meet at K, find GK : EH.

13. DEF is a right-angled △, E being the right angle. G is taken in DE produced such that DG : GF = DF : EF. Prove that ∠ DFG is right

14. If the perpendicular to the base of a △ from the vertex be a mean proportional to the segments of the base, the triangle is right angled.

15. DGH is any △, and from K the middle point of GH a line is drawn cutting DH at E and GD produced at F. Prove GF : FD = HE : ED. Prove the converse also.

16. AD and AE are the interior and exterior bisectors of the vertical angle of △ ABC meeting the base at D and E. Through C, FCG is drawn || to AB meeting AD and AE at F and G. Prove that FC = CG.

17. HKL is an isosceles △, having HK = HL; KL is produced to D and DEF is drawn cutting HL at E, and HK at F. Prove DE : DF = EL : KF

18. DP and DQ are perpendiculars to the bisectors of the interior angles E and F of any △ DEF. Prove PQ || EF.

19. PX and QY are perpendiculars from P and Q to XY; PY and QX intersect at R, and RZ is perpendicular to XY. Prove ∠ PZX = ∠ QZY.

20. ABC is any △, and AD is taken along AC such that AC : AB = AB : AD; also CF is taken along AC such that AC : CB = CB : CF. Prove BF = BD.

21. The perpendicular KD to the hypotenuse HL of a right-angled △ KHL is produced to E such that KD : DH = DH : DE. Prove HE || KL.

22. DEF is a $\triangle$ inscribed in a circle, and P and Q are taken in DE and DF such that DP : PE = DQ : QF. Show that the circle described about D, P, Q touches the given circle at D.

23. D is a point in LM a side of $\triangle$ LMN, DE is $\parallel$ to MN and EF $\parallel$ to LM, meeting the sides at E and F. Prove LD : DM = MF : FN.

24. A variable line through a fixed point O meets two $\parallel$ st. lines at P and Q. Prove OP : OQ a constant ratio.

25. If the nonparallel sides of a trapezium are cut in the same ratio by a st. line, show that this line is $\parallel$ to the $\parallel$ sides.

26. ABCDE is a polygon, O a point within it. If X, Y, Z, P, Q are points in OA, OB, OC, OD, OE such that OX : OA = OY : OB = etc., show that the sides of XYZPQ are $\parallel$ to those of ABCDE.

27. DE is a st. line, F any point in it; find a point P in DE produced such that PD : PE = DF : FE.

28. St. lines PD, PE, PF and PG are such that each of the $\angle$s DPE, EPF, FPG is equal to half a right angle. DEFG cuts them such that PD = PG. Prove that DG : FG = FG : EF.

29. GH is a chord of a circle, K and D points on the two arcs respectively; KH and KD are joined and GD meets KH produced at E; EF $\parallel$ to GH meets KD produced. Show that EF is equal to the tangent from F.

30. DEF, DEG are two circles, the centre P of DEG being on the circumference of DEF. A st. line PHGF cuts the common chord at H. Prove that PH : PG = PG : PF.

31. EF is the diameter of a circle. PQ is a chord $\perp$ to EF, a chord QXR cuts EF at X, and PR, EF produced meet at Y. Show that EX : EY = FX : FY.

32. O is a fixed point and P a variable point on the circumference of a circle; PO is produced to Q such that $OQ:OP=m:n$. Find the locus of Q.

33. LMN is a △ inscribed in a circle, ∠ L is bisected by LED cutting MN at E and the arc at D. Prove △s LEN and LMD similar.

34. The ∠ D of the △ DEF is bisected by DP cutting EF in P; QPR is ⊥ to DP meeting DE and DF at Q and R; RS is ‖ to EF meeting DE at S. Prove SE = EQ.

35. AOB, COD and EOF are any three st. lines; ACE is ‖ to FDB. Prove AC : CE = BD : DF. State and prove a converse to this theorem.

36. Two circles DEF and DEG intersect; a tangent DF is drawn to DEG, and EG to DEF. Show that DE is a mean proportional between FE and DG.

37. EFGH is a quadrilateral, the diagonals EG and FH meet at Q. Prove △ EFH : △ FGH = EQ : QG.

38. EFGH is a quadrilateral of which the sides EH and FG produced meet at P. Prove △ EFG : △ FGH = EP : PH.

39. G is the middle point of the st. line MN, PE a st. line ‖ to MN. Any st. line EFGH cuts PN at F and PM produced at H. Prove EF : FG = EH : HG.

40. ABC is a △ having ∠ B = ∠ C = twice ∠ A, BD bisects the ∠ B meeting AC at D. Prove AC : AD = AD : DC; also prove △ ABC : △ ABD = △ ABD : △ BDC.

41. EFGH is a cyclic quadrilateral, EG and FH intersect at O, and OP and OQ are ⊥s to EH and FG. Show that OP : OQ = EH : FG.

42. EF is the diameter of a circle and P and Q any points on the circumference on opposite sides of EF; QR is ⊥ to EF meeting EP at S. Prove △ ESQ ||| △ EQP.

43. **ABC** is a $\triangle$ inscribed in a circle, centre **O**, **AD** a $\perp$ to **BC**, **AOE** a diameter. Prove $\triangle$s **ADC** and **ABE** similar: and **AD.AE** = **AB.AC**.

44. **EFG** is a $\triangle$ inscribed in a circle, **ED** $\parallel$ to the tangent at **G** meets the base at **D**. Prove that **FG : FE = EG : ED**.

45. Find the ratio of the segments of the hypotenuse of a right-$\angle$d $\triangle$ made by a perpendicular on it from the vertex, if the ratio of the sides be (1) 1 : 2; (2) $m : n$

46. **PQ** is the diameter of a circle; a tangent is drawn from a point **R** on the circumference, **PS** and **QT** are $\perp$ to the tangent. Prove $\triangle$s **PRQ**, **RPS** and **RTQ** similar; also show that $\triangle$ **PRQ** is half of **PSTQ**.

47. **PQ** and **PR** are tangents to a circle, **PST** is a secant meeting the circle at **S** and **T**. Prove **QT : QS = RT : RS**.

48. Two circles intersect at **E** and **F**; from **P**, any point on one of them, chords **PED**, **PFG** are drawn, **EF** and **DG** meet at **Q** and **PQ** cuts the circle **PEF** at **R**. Prove **R, F, G, Q** concyclic; also that PQ^2 is equal to the sum of the squares on the tangents to the circle **EFGD** from **P** and **Q**.

49. **PBR** is a st. line, and similar segments of circles, **PAB** and **BAR**, are described on **PB** and **BR** and on the same side of **PR**. **PAC** and **RAD** are drawn to meet the circles at **C** and **D**. Prove **PD : RC = PB : BR**.

NOTE.—*Segments of circles are said to be similar when they contain equal angles.*

50. **PMQ** is the diameter of a circle **PRQ**, **PX** and **QY** are $\parallel$ tangents, **XRY** is any other tangent, **PY** and **XQ** meet at **O**. Show that **RO** is $\parallel$ to **PX**; that **RO** produced to **M** is $\perp$ to the diameter; and that **MO = OR**.

51. **ABCD** is a rectangle, a st. line **APQR** is drawn cutting **BC** at **P**, the circle circumscribing the rectangle at **Q** and **DC** produced at **R**, and such that **AC** bisects $\angle$ **DAR**. Prove **DC : CR = PQ : PA**.

52. PQRS is a square. A st. line PFED cuts QS at F, SR at E and QR produced at D. Prove FR a tangent to the circle described about DER; also that EF : PF = PF : FD.

53. FGHK is a cyclic quadrilateral, the ∠ GFE is made equal to ∠ HFK and E is in GK. Prove △s FEK and FGH similar

54. PA and PB are tangents to a circle, centre O, AB meets PO in R; PCD is any secant, OS is ⊥ to PD, and AB and OS produced meet at Q. Prove (1) P, R, S, Q concyclic; (2) $PO.OR = OA^2$; (3) QD and QC are tangents to the given circle.

55. DEF is a △ and P and Q are points in ED and FE such that EP : PD = FQ : QE, and PQ meets DF produced at R. Prove $RF : RD = PE^2 : PD^2$. (*Through* F *draw a st. line* ∥ *to* DE *to meet* PR.)

56. If a square be inscribed in a rt.-∠d △ having one side on the hypotenuse, show that the three segments of the base are in continued proportion.

57. FGH is a △ and ∠ G and ∠ H are bisected by st. lines which cut the opposite sides at D and E; if DE is ∥ to GH, then FG = FH.

58. From P, the middle point of an arc of a circle cut off by a chord QR, any chord PDE is drawn cutting QR at D. Show that $PQ^2 = PD.PE$.

59. Draw a st. line through a given point so that the perpendiculars on it from two other given points may be (1) equal, (2) one twice the other, (3) three times the other, (4) in a given ratio.

60. LMN is an isosceles △, the base MN is produced both ways, in NM produced any point P is taken, and in MN produced NQ is taken a third proportional to PM and LM. Prove △s PLQ and PLM similar.

61. **EDOF** is the diameter of a circle, centre **O**. **PE** and **PG** are tangents to the circle; **GD** is $\perp$ to **EF**. Prove **GD : DE = OE : EP**.

62. **DEF** is a $\triangle$ inscribed in a circle, centre **O**. The diameter $\perp$ to **EF** cuts **DE** at **P** and **FD** produced at **Q**. Prove $\triangle$s **EPO** and **FOQ** similar; and hence $OE^2 = OP.OQ$.

63. **ABC** is a $\triangle$ inscribed in a circle. The exterior $\angle$ at **A** is bisected by a st. line cutting **BC** produced at **D** and the circumference at **E**. Prove **BA.AC = EA.AD**.

64. **EFGH** is a cyclic quadrilateral, **P** a point on the circumference, **PQ**, **PR**, **PS**, **PT** are $\perp$ to **EF**, **FG**, **GH**, **HE** respectively. Prove $\triangle$s **PTQ** and **PSR** similar; and **PT.PR = PS.PQ**.

65. Any three || chords **AB**, **CD**, **EF** are drawn in a circle, **AC** and **BD** meet **EF** produced at **Q** and **R**, **P** is a point in the arc **EF**, and **PA** and **PD** meet **EF** at **M** and **N**. Prove $\triangle$s **AQM** and **NDR** similar; hence show that, for all positions of **P**, **QM.NR** is constant.

66. Two tangents **TMP** and **TNQ** are drawn to a circle, centre **O**, and the st. line **POQ** is $\perp$ to **TO**. **MN** is any other tangent to the circle. Prove $\triangle$s **MPO** and **NQO** similar.

67. **DH** is a median of the $\triangle$ **DEF**, **PQ** is || to **EF** cutting **DE** at **P** and **DF** at **Q**. Show that **PF** and **EQ** intersect on **DH**.

68. **LNM** is a $\triangle$ inscribed in a semicircle, diameter **LM**. **NM** is greater than **NL**. On opposite sides of **LN** the $\angle$ **LNP** is made equal to $\angle$ **LNQ**, **P** and **Q** lying along **LM**. Prove **PL : LQ = PM : QM**.

69. **EFGH** is a ||gm, and **RS** is drawn || to **HF** meeting **EH** and **EF** at **R** and **S**. Show that **RG** and **SG** cut off equal segments of the diagonal **FH**. Prove a converse of this.

70. ABC is a △ and AB, AC are produced to D, E so that BD = CE; DE and BC produced meet at F. Show that AD : AE = FC : FB.

71. Two circles, having centres O, P, intersect, the centre O being on the circumference of the other circle. GDE touches the circle with centre O at G and cuts the other at D, E, and EPF is a diameter. Prove △ OGD ||| △ OEF; and hence, that OD.OE is constant for all positions of the tangent.

72. Two circles touch externally at P; EF a chord of one circle touches the other at D. Prove PE : PF = ED : DF.

73. EOF is the diameter of a circle, with centre O, DP any chord cutting the diameter; OSQR ⊥ to DP meets DP at S, DE at Q, and PE at R. Prove △s EDF and RSP similar; also $OQ.OR = OD^2$.

74. Divide an arc of a circle into two parts so that the chords which cut them off shall have a given ratio to each other.

75. LMN is a △, and XY ∥ MN meets LM at X and LN at Y; MN is produced to D so that ND = XY, and XP ∥ to LD meets MN at P. Prove MN : ND = ND : NP.

76. Two circles intersect and a st. line CDOEF cuts the circumferences at C, D, E, F and the common chord at O. Show that CD : DO = EF : OE.

77. DX ⊥ EF and EY ⊥ DF in △ DEF. The lines DX, EY cut at O. Prove that EX : XO = DX : XF.

78. From a point P without a circle two secants PKL, PMN are drawn to meet the circle in K, L, M, N. The bisector of ∠ KPM meets the chord KM at E and the chord LN at F. Prove that LF : FN = ME : EK.

79. QR is a chord ∥ to the tangent at P to a circle. A chord PD cuts QR at E. Prove that PQ is a mean proportional between PE and PD.

80. **DEF**, **DEG** are two fixed circles and **FEG** is a st. line. Show that the ratio **FD : DG** is constant for all positions of the st. line **FEG**.

81. **DEF** is a st. line, and **EG**, **FH** are any two || st. lines on the same side of **DEF** such that **EG : FH = DE : DF**. Prove that **D**, **G**, **H** are in a st. line.

82. From a given point on the circumference of a circle draw two chords which are in a given ratio and contain a given ∠.

83. **DEF** is a △ and on **DE**, **DF** two △s **DLE**, **DFM** are described externally such that ∠ **FDM** = ∠ **EDL** and ∠ **DFM** = ∠ **DLE**. Prove △ **DLF** ||| △ **DEM**.

84. **DEFG** is a ||gm and **P** is any point in the diagonal **EG**. The st. line **KPL** meets **DE** at **K** and **FG** at **L**, and **MPN** meets **EF** at **M** and **GD** at **N**. Prove **KM** || **NL**.

85. **ABCD** is a ||gm and **PQ** is a st. line || **AB**. The st. lines **PA**, **QB** meet at **R** and **PD**, **QC** meet at **S**. Prove **RS** || **AD**.

86. If the three sides of one △ are respectively ⊥ to the three sides of another △, the two △s are similar.

87. Find a point whose ⊥ distances from the three sides of a △ are in the ratio 1 : 2 : 3.

88. Squares are described each with one side on one given st. line and one vertex on another given st. line. Find the locus of the vertices which are on neither.

89. If the sides of a rt.-∠d △ are in the ratio 3 : 2, prove that the ⊥ from the vertex of the rt. ∠ to the hypotenuse divides it in the ratio 9 : 4.

90. **HK** is a diameter of a circle and **L** is any point on the circumference. A st. line ⊥ **HK** meets **HK** at **D**, **HL** at **E**, **KL** at **G**, and the circumference at **F**. Show that $DF^2 = DE.DG$.

91. The st. line joining a fixed point to any point on the circumference of a given circle is divided in a given ratio at **P**. Prove that the locus of **P** is a circle.

92. **DEFG** is a quadrilateral and **P, Q, R, S** are points on **DE, EF, FG, GD** such that $DP : DE = FQ : FE = FR : FG = DS : DG$. Prove that **PQRS** is a ||gm.

93. **DEFG** is a ||gm, and a line is drawn from **E** cutting **DF** in **P**, **DG** in **Q** and **FG** produced in **R**. Prove that $PQ : PR = DP^2 : PF^2$; and that $PQ.PR = EP^2$.

94. If $\triangle\ DEF : \triangle\ GHK = DE.EF : GH.HK$, prove that $\angle$s **E, H** are either equal or supplementary.

95. From a point **P** without a circle draw a secant **PQR**, such that **QR** is a mean proportional between **PQ** and **PR**.

96. Through a point of intersection of two circles draw a line such that the chords intercepted by the circles are in a given ratio.

97. If two $\triangle$s are on equal bases and between the same ||s, the intercepts made by the sides of the $\triangle$s on any st. line || to the base are equal.

98. The radius of a fixed circle is 38 mm., and a chord **LM** of the circle is divided at **P** such that $LP.PM = 225$ sq. mm. Construct the locus of **P**.

99. If the tangents from a given point to any number of intersecting circles are all equal, all the common chords of the circles pass through that point.

100. Circles are described passing through two fixed points; find the locus of a point from which the tangents to all the circles are equal.

101. **DEF** is a $\triangle$ having $\angle$ **E** a rt. $\angle$. A circle is described with centre **D** and radius **DE**; from **F** a secant is drawn cutting the circle at **G, H**; and **EX** is drawn $\perp$ **DF.** Show that **D, X, G, H** are concyclic.

102. **GD** is a chord drawn || to the diameter **LM** of a circle. **LG**, **LD** cut the tangent at **M** at **E**, **F** respectively. Prove that $LG.GE + LD.DF = LM^2$.

103. **LM** is a diameter of a circle, and on the tangent at **L** equal distances **LP**, **PQ** are cut off. **MP**, **MQ** cut the circumference at **R**, **S** respectively. Prove that **LR : RS = LM : MS.**

104. **GH** drawn in the △ **DEF** meets **DE** in **G** and **DF** in **H.** From **D** any line **DLK** is drawn cutting **GH** in **L** and **EF** in **K.** From **L** the st. lines **LM, LN** are drawn || **KH, KG** and meeting **DH, DG** at **M, N** respectively. Prove △ **LMN** ||| △ **KHG.**

105. In a given △ inscribe an equilateral △ so as to have one side || to a side of the given △.

106. In a given △ **DEF** draw a st. line **PQ** || **ED** meeting **EF** in **P** and **DF** in **Q,** so that **PQ** is a mean proportional between **EP** and **PF.**

107. Two circles intersect at **E, F,** and **DEG** is the st. line ⊥ **EF** and terminated in the circumferences. **HEK** is any other st. line through **E** terminated in the circumferences. **HF, DF, KF, GF** are drawn. Prove, by similar △s, that **DG** > **HK**

108. In △ **ABC** the bisectors of ∠ **A** and of the exterior ∠ at **A** meet the st. line **BC** at **D** and **E.** Show that $DE = \frac{2\,abc}{c^2 - b^2}$.

109. If two circles intercept equal chords **PQ, RS** on any st. line, the tangents **PT, RT** to the circles at **P, R** are to one another as the diameters of the circles.

110. **DEF** is a △ having **DF** > **DE.** From **DF** a part **DG** is cut off equal to **DE,** and **GH** is drawn || **DE** to meet

EF at **H**. From **GF** a part **GK** is cut off equal to **GH**, and **KL** is drawn || **GH** to meet **EF** at **L**; etc. Prove that **DE**, **GH**, **KL**, etc., are in continued proportion.

111. A circle **P** touches a circle **Q** internally, and also touches two || chords of **Q**. Prove that the $\perp$ from the centre of **P** on the diameter of **Q** which bisects the chords is a mean proportional between the two extremes of the three segments into which the diameter is divided by the chords.

112. **PX** is the $\perp$ from a point **P** on the circumference of a circle to a chord **QR**, and **QY**, **RZ** are $\perp$s to the tangent at **P**. Prove that $PX^2 = QY.RZ$.

113. Prove, by using 112, that if $\perp$s are drawn to the sides and diagonals of a cyclic quadrilateral from a point on the circumference of the circumscribed circle, the rectangle contained by the $\perp$s on the diagonals is equal to the rectangle contained by the $\perp$s on either pair of opposite sides.

114. The projections of two || st. lines on a given st. line are proportional to the st. lines.

115. **DEFG** is a square, and **P** is a point in **GF** such that $DP = FP + FE$. Prove that the st. line from **D** to the middle point of **EF** bisects $\angle$ **PDE**.

116. **DEF**, **GEF** are $\triangle$s on opposite sides of **EF**, and **DG** cuts **EF** at **H**. Prove that $\triangle DEF : \triangle GEF = DH : HG$.

117. From the intersection of the diagonals of a cyclic quadrilateral $\perp$s are drawn to a pair of opposite sides: prove that these $\perp$s are in the same ratio as the sides to which they are drawn.

118. **P**, **Q**, **R**, **S** are points in a st. line, $PX \parallel QY$, $RX \parallel SY$, and **XY** meets **PS** at **O**. Prove that $OP.OS = OQ.OR$.

119. From a point **T** without a circle tangents **TP**, **TQ** and a secant **TRS** are drawn. Prove that in the quadrilateral **PRQS** the rect. $PR.QS =$ the rect. $RQ.SP$.

Miscellaneous Exercises

1. Draw four circles each of radius $1\frac{3}{4}$ inches, touching a fixed circle of radius 1 inch and also touching a st. line $1\frac{1}{2}$ inches distant from the centre of the circle.

2. **DE**, **FG** are || chords of the circle **DEGF**. Prove that $DE.FG = DG^2 - DF^2$.

3. If two circles touch externally at **A** and are touched at **B**, **C** by a st. line, the st. line **BC** subtends a rt. $\angle$ at **A**.

4. Of all $\triangle$s of given base and vertical $\angle$, the isosceles $\triangle$ has the greatest area.

5. **ABC** is an equilateral $\triangle$ inscribed in a circle, **P** is any point on the circumference. Of the three st. lines **PA**, **PB**, **PC**, shew that one equals the sum of the other two.

6. Construct a rt.-$\angle$d $\triangle$, given the radius of the inscribed circle and an acute $\angle$ of the $\triangle$.

7. The diagonals **AC**, **BD** of a cyclic quadrilateral **ABCD** cut at **E**. Show that the tangent at **E** to the circle circumscribed about $\triangle$ **ABE** is || to **CD**.

8. **A**, **B**, **C** are three points on a circle. The bisector of $\angle$ **ABC** meets the circle again at **D**. **DE** is drawn || to **AB** and meets the circle again at **E**. Show that **DE** = **BC**.

9. The side of an equilateral $\triangle$ circumscribed about a circle is double the side of the equilateral $\triangle$ inscribed in the same circle.

10. **AB** is the diameter of a circle and **CD** a chord. **EF** is the projection of **AB** on **CD**. Show that **CE** = **DF**.

11. Construct an isosceles $\triangle$, given the base and the radius of the inscribed circle.

12. Two circles touch externally. Find the locus of the points from which tangents drawn to the circles are equal to each other.

13. Two circles, centres C, D, intersect at A, B. PAQ is a st. line cutting the circles at P, Q. PC, QD intersect at R. Find the locus of R

14. Two circles touch internally at A; BC, a chord of the outer circle, touches the inner circle at D. Show that AD bisects ∠ BAC.

15. P is a given point on the circumference of a circle, of which AB is a given chord. Through P draw a chord PQ that is bisected by AB.

16. On a given base construct a △ having given the vertical ∠ and the ratio of the two sides.

17. AB is a given st. line and P, Q are two points such that AP : PB = AQ : QB. Show that the bisectors of ∠s APB, AQB cut AB at the same point.

18. AB is a given st. line and P, Q are two points such that AP : PB = AQ : QB. Show that the bisectors of the exterior ∠s at P, Q of the △s APB, AQB meet AB produced at the same point.

19. AB is a given st. line and P is a point which moves so that the ratio AP : PB is constant. The bisectors of the interior and exterior ∠s at P of the △ APB, meet AB and AB produced at C, D respectively. Show that the locus of P is a circle on CD as diameter.

20. AB is a st. line 2 inches in length. P is a point such that AP is twice BP. Construct the locus of P.

21. Two circles touch externally, and A, B are the points of contact of a common tangent. Show that AB is a mean proportional between their diameters.

22. If on equal chords segments of circles are described containing equal $\angle$s, the circles are equal.

23. Construct a quadrilateral such that the bisectors of the opposite $\angle$s meet on the diagonals.

24. Draw a circle to pass through a given point and touch two given st. lines.

25. Draw a circle to touch a given circle and two given st. lines.

26. Draw a circle to pass through two given points and touch a given circle.

27. Construct a rt.-$\angle$d $\triangle$ given the hypotenuse and the radius of the inscribed circle.

28. In $\triangle$ ABC the inscribed circle touches AB, AC at D, E respectively. The line joining A to the centre cuts the circle at F. Show that F is the centre of the inscribed circle of $\triangle$ ADE.

29. The inscribed circle of the rt.-$\angle$d $\triangle$ ABC touches the hypotenuse BC at D. Show that rect. BD.DC = $\triangle$ ABC.

30. If on the sides of any $\triangle$ equilateral $\triangle$s are described outwardly, the centres of the circumscribed circles of the three equilateral $\triangle$s are the vertices of an equilateral $\triangle$.

31. Describe three circles to touch one another externally and a given circle internally.

32. Show that two circles can be described with the middle point of the hypotenuse of a rt.-$\angle$d $\triangle$ as centre to touch the two circles described on the two sides as diameters.

33. A st. line AB of fixed length moves so as to be constantly $\parallel$ to a given st. line and A to be on the circumference of a given circle. Show that the locus of B is an equal circle.

34. Construct an isosceles $\triangle$ equal in area to a given $\triangle$ and having the vertical $\angle$ equal to one of the $\angle$s of the given $\triangle$.

35. If two chords AB, AC, drawn from a point A in the circumference of the circle ABC, are produced to meet the tangent at the other extremity of the diameter through A in D, E respectively, then the $\triangle$ AED is similar to $\triangle$ ABC.

36. If a st. line be divided into two parts, the sq. on the st. line equals the sum of the rectangles contained by the st. line and the two parts.

37. ABCD is a quadrilateral inscribed in a circle. AB, DC meet at E and BC, AD meet at F. Show that the sq. on EF equals the sum of the sqs. on the tangents drawn from E, F to the circle.

38. The st. line AB is divided at C so that AC = 3 CB. Circles are described on AC, CB as diameters and a common tangent meets AB produced at D. Show that BD equals the radius of the smaller circle.

39. DE is a diameter of a circle and A is any point on the circumference. The tangent at A meets the tangents at D, E at B, C respectively. BE, CD meet at F. Show that AF is $\parallel$ to BD.

40. TA, TB are tangents to a circle of which C is the centre. AD is $\perp$ BC. Show that TB : BC = BD : DA.

41. ABCD is a quadrilateral inscribed in a circle. BA, CD produced meet at P, and AD, BC produced meet at Q. Show that PC : PB = QA : QB.

42. Divide a given arc of a circle into two parts so that the chords of these parts shall be to each other in the ratio of two given st. lines.

43. Describe a circle to pass through a given point and touch a given st. line and a given circle.

44. LMN is a rt.-∠d △ with L the rt. ∠. On the three sides equilateral △s LEM, MFN, NDL are described outwardly. LG is ⊥ MN. Prove that △ FGM = △ LEM and △ FGN = △ NDL.

45. L is the rt. ∠ of a rt.-∠d △ LMN in which LN = 2 LM. Also LX ⊥ MN. Prove that LX = $\frac{2}{5}$ MN.

46. A st. line meets two intersecting circles in P and Q, R and S and their common chord in O. Prove that OP, OQ, OR, OS, taken in a certain order, are proportionals.

47. LMN is a semi-circle of which O is the centre, and OM ⊥ LN. A chord LDE cuts OM at D. Prove that LM is a tangent to the circle MDE.

48. The bisector of ∠ F of △ FGH meets the base GH in E and the circumcircle in D. Prove that $DG^2 = DE.DF$.

49. POQ, ROS are two st. lines such that PO : OQ = 3 : 4 and RO : OS = 2 : 5. Compare areas of △s POR, QOS; and also areas of △s POS, QOR.

50. Trisect a given square by st. lines drawn ∥ to one of its diagonals.

51. Construct a △ having its base 8 cm., the other sides in the ratio of 3 to 2, and the vertical ∠ = 75°.

52. In two similar △s, the parts lying within the △ of the right bisectors of corresponding sides have the same ratio as the corresponding sides of the △.

53. KMN, LMN are △s on the same base and between the same ∥s. KN, LM cut at E. A line through E, ∥ MN, meets KM in F and LN in G. Prove that FE = EG.

54. Construct a △ having given the vertical ∠, the ratio of the sides containing that ∠, and the altitude drawn to the base.

55. From a point P without a circle two secants PFG, PED are drawn, and PQ drawn || FD meets GE produced at Q. Prove that PQ is a mean proportional between QE, QG.

56. LD bisects ∠ L of △ LMN and meets MN at D. From D the line DE || LM meets LN at E, and DF || LN meets LM at F. Prove that FM : EN = $LM^2 : LN^2$.

57. LMN is a △ ∠ rt.-d at L. LD ⊥ MN and meets a line drawn from M ⊥ LM at E. Prove that △ LMD is a mean proportional between △s LDN, MDE.

58. Two circles touch externally at D and PQ is a common tangent. PD and QD produced meet the circumferences at L, M respectively. Show that PM and QL are diameters of the circles.

59. The common tangent to two circles which intersect subtends supplementary ∠s at the points of intersection.

60. Two circles intersect at Q and R, and ST is a common tangent. Show that the circles described about △s STR, STQ are equal.

61. A st. line DEF is drawn from D the extremity of a diameter of a circle cutting the circumference at E and a fixed st. line ⊥ to the diameter at F. Show that the rect. DE.DF is constant for all positions of DEF.

62. A chord LM of a circle is produced to E such that ME is one-third of LM; a tangent EP is drawn to the circle and produced to D such that PD = EP. Prove that △ ELD is isosceles.

63. Draw a st. line to touch one circle and to cut another, the chord cut off being equal to a given st. line.

64. Two equal circles are placed so that the transverse common tangent is equal to the radius. Show that the tangent from the centre of one circle to the other equals the diameter of each circle.

65. Construct a △ having its medians respectively equal to three given st. lines.

66. Construct a △ given one side and the lengths of the medians drawn from the ends of that side.

67. Construct a △ given one side, the median drawn to the middle point of that side, and a median drawn from one end of that side.

68. Construct a △ having $\angle$ A = 20°, $\angle$ C = 90°, and $c - a = 4$ cm.

69. Construct a △ having $\angle$ C = 90°, $b = 6$ cm., and $c - a = 3{\cdot}5$ cm.

70. Construct a △ having $a = 7$ cm., $c - b = 3$ cm., and $\angle$ C − $\angle$ B = 28°.

71. If a st. line be drawn in any direction from one vertex of a ||gm, the ⊥ to it from the opposite vertex equals the sum or difference of the ⊥s to it from the two remaining vertices.

72. PQ is a chord of a circle ⊥ to the diameter LM, and E is any point in LM. If PE, QE meet the circumference in S, R respectively, show that PS = QR; and that RS ⊥ LM.

73. P is any point in a diameter LM of a circle, and QR is a chord ∥ LM. Prove that $PQ^2 + PR^2 = PL^2 + PM^2$.

74. On the hypotenuse EF of the rt.-$\angle$d △ DEF a △ GEF is described outwardly having $\angle$GEF = $\angle$DEF and $\angle$GFE a rt. $\angle$. Prove that △ GFE : △ DEF = GE : ED.

75. Two quadrilaterals whose diagonals intersect at equal $\angle$s are to one another in the ratio of the rectangles contained by the diagonals.

76. P is any point in the side LM of a $\triangle$ LMN. The st. line MQ, $\parallel$ PN, meets LN produced at Q; and X, Y are points in LM, LQ respectively, such that $LX^2 = LP.LM$ and $LY^2 = LN.LQ$. Prove that $\triangle$ LXY = $\triangle$ LMN.

77. EFP, EFQ are circles and PFQ is a st. line. ER is a diameter of circle EFP and ES a diameter of EFQ. Prove $\triangle$ EPR : $\triangle$ EQS as the squares on the radii of the circles.

78. If P be the point of intersection of an external common tangent PQR to two circles with the line of centres, prove that PQ : PR as the radii of the circles. Also, if PCDEF is a secant, prove that PC : PE = PD : PF.

79. A point E is taken within a quadrilateral FGHK such that $\angle$EFK = $\angle$GFH and $\angle$EKF = $\angle$GHF. GE is joined. Prove $\triangle$ FEG ||| $\triangle$ FHK.

80. Through a given point within a circle, draw a chord that is divided at the point in a given ratio.

81. From P, a point on the circumference of a circle, tangents PE, PF are drawn to an inner concentric circle. GEFH is a chord, and PE meets the circumference at Q. Prove $\triangle$s PGF, PEH, GEQ similar; also show that $GQ^2 : GP^2 = GE : GF$.

82. L is the vertex of an isosceles $\triangle$ LMN inscribed in a circle, LRS is a st. line which cuts the base in R and meets the circle in S. Prove that $SL.RL = LM^2$.

83. PQR is a rt.-$\angle$d $\triangle$ with P the rt. $\angle$. PD $\perp$ QR; DM $\perp$ PQ and DN $\perp$ PR. Prove that $\angle$ QMR = $\angle$ QNR.

84. DEF is an isosceles $\triangle$ with $\angle$ D = 120.° Show that if EF be trisected at G and H, the $\triangle$ DGH is equilateral.

85. **AS** and **AT**, **BP** and **BQ** are tangents from two points **A** and **B** to a circle. **C**, **D**, **E**, **F** are the middle points of **AS**, **AT**, **BP**, **BQ** respectively. Prove that **CD**, **EF**, produced if necessary, meet on the right bisector of **AB**. *(Let* **O** *be the centre of the circle;* **L** *and* **M** *the points where* **OA**, **OB** *cut the chords of contact. Prove* **A**, **L**, **M**, **B** *concyclic, etc.)*

86. If from the middle point of an arc two st. lines are drawn cutting the chord of the arc and the circumference, the four points of intersection are concyclic.

87. If a st. line be divided at two given points, find a third point in the line, such that its distances from the ends of the line may be proportional to its distances from the two given points.

88. Prove geometrically that the arithmetic mean between two given st. lines is greater than the geometric mean between the two st. lines.

89. A square is inscribed in a rt.-angled triangle, one side of the square coinciding with the hypotenuse; prove that the area of the square is equal to the rectangle contained by the extreme segments of the hypotenuse.

90. Any regular polygon inscribed in a circle is the geometric mean between the inscribed and circumscribed regular polygons of half the number of sides.

91. The diagonal and the diagonals of the complements of the parallelograms about the diagonal of a parallelogram are concurrent.

92. Develop the formula for the area of a $\triangle$, $\sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$ and a, b, c are the sides.

Solution of 92. In $\triangle$ **ABC**, draw **AX** $\perp$ **BC**, and let **AX** $= h$, **BX** $= x$. Then **CX** $= a - x$.

Area of $\triangle$ **ABC** $= \frac{1}{2} a h$.

$$h^2 = b^2 - (a - x)^2 = c^2 - x^2,$$

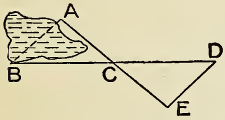
$$\therefore \quad x = \frac{a^2 - b^2 + c^2}{2a}.$$

$$h^2 = c^2 - \frac{(a^2 - b^2 + c^2)^2}{4a^2}.$$

$$\begin{aligned} 4a^2 h^2 &= 4a^2 c^2 - (a^2 - b^2 + c^2)^2 \\ &= (2ac + a^2 - b^2 + c^2)(2ac - a^2 + b^2 - c^2) \\ &= \{(a + c)^2 - b^2\}\{b^2 - (a - c)^2\} \\ &= (a + b + c)(a - b + c)(a + b - c)(b - a + c) \\ &= 2s(2s - 2b)(2s - 2c)(2s - 2a). \end{aligned}$$

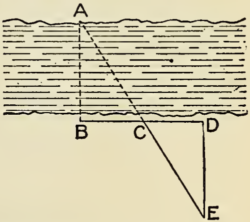
$$\therefore \tfrac{1}{4} a^2 h^2 = s(s - a)(s - b)(s - c),$$

$$\text{And } \tfrac{1}{2} ah = \sqrt{s(s - a)(s - b)(s - c)}.$$



93. Show from the diagram how the distance between two points, **A, B** at opposite sides of a pond may be found by measurements on land.

94. Show from the diagram how the breadth of a river may be found by measurements made on one side of it.



95. Given a st. line **AB**, construct a continuation of it **CD**, **AB** and **CD** being separated by an obstacle.

96. **AB**, **CD** are two lines which would meet off the paper. Draw a st. line which would pass through the point of intersection of **AB**, **CD**, and bisect the $\angle$ between them.

INDEX TO DEFINITIONS

Geometry

1 A point is that which has position but not magnitude.

2 A line is that which has length but niether breadth nor thickness.

3 A St. line is the shortest distance between two points.

4 A curved line changes in direction continually from point to point.

5. A broken line is a line composed of different successive st. lines.

6 Surface is that which has length and breadth but no thickness.

7 Plane Surface is that which has any two points taken, the st. line joining them lies wholly in that surfake.

8. Geometry is the science which investigates the properties of figure and relation of figures to one another.

9 Solid is that which has length, width and thickness.

10 Any combination of points, lines, surfaces and solids is called a figure.

11. Plane Geometry: the figure considered in each propositions are confined to one plane while Solid Geometry treats of figure the parts of figures which are not all in the same place.

Measurements of St. lines: It is required to measure the the distance between two points A + B.

Vertically opposite ∠'s
The angles that are opposite each other when two St lines cross.

AB > greater than AC.
AB ∠ less " AC.

when 2 angles have same vertex and a common arm and remaining arms on opposite sides of common arm they are said to be <u>adjacent angles</u>.

When 1 st. line stands on another so as to make the adjacent angles equal to one another each of the angles is called a right angle and each ~~(angle)~~ line is said to be <u>perpendicular</u> to each other.

<u>Triangles are equal if they have</u>:

(1). Two sides and the contained ∠ of one equal respectively to two sides and the contained ∠ of the other. (I—1)

(2) 3 sides of one respectively = to 3 sides of the other (<u>III</u>—1)

(3) 2 ∠'s and 1 side respectively equal to 2 ∠'s & 1 side of the other. (<u>IV</u>—I)

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